Analysis Qualifying Examination 1 **August 16, 2021**

Please start each problem on a new page and remember to write your code on each page of your answers.

You should exercise good judgement in deciding what constitutes an adequate solution. In particular, you should not try to solve a problem by just quoting a theorem that reduces what you are asked to prove to a triviality. If you are not sure whether you may use a particular theorem, ask the proctor.

[17] **1.** Suppose μ_0 is a finite premeasure on an algebra $\mathscr{A} \subseteq \mathscr{P}(X)$, where $\mathscr{P}(X)$ is the power set of X. Let $\mu^*\colon \mathscr{P}(X) \to [0,\infty]$ be the outer measure induced by μ_0, and let $\mathscr{M}^*$ be the collection of μ^*-measurable subsets of X. Let $E \subseteq X$. Prove that the following are equivalent.

(a) $E \in \mathscr{M}^*$.

(b) $\mu^*(E) + \mu^*(X \setminus E) = \mu_0(X)$.

[8] **2. (a)** Let $f\colon \mathbf{R} \to \mathbf{R}$ be increasing. Show that f is Borel-measurable.

[8] **(b)** Let $f\colon \mathbf{R} \to \mathbf{R}$ be continuous and strictly increasing. Show that f maps Borel sets to Borel sets.

[17] **3.** Let $(X, \mathscr{A}, \mu)$ be a measure space. Let $f, f_1, f_2, \ldots\colon X \to \mathbf{R}$ be measurable. Suppose $f_n \to f$ pointwise and $\int |f_n|\, d\mu \to \int |f|\, d\mu < \infty$. Prove that $\int |f - f_n|\, d\mu \to 0$.

[17] **4.** Let $F\colon \mathbf{R} \to \mathbf{R}$ be absolutely continuous on each compact interval and increasing. Let μ be the Lebesgue-Stieltjes measure on $\mathbf{R}$ corresponding to F and let m be Lebesgue measure on $\mathbf{R}$. Prove that for each Borel set $E \subseteq \mathbf{R}$, we have $\mu(E) = \int_E F'\, dm$.

[16] **5.** Let (X, ρ) be a complete metric space and let $f\colon X \to \mathbf{R}$ be lower semicontinuous (which means that for each $y \in \mathbf{R}$, the set $\{ f > y \}$ is open in X). Let G be the set of all $x \in X$ such that f is bounded above in some neighborhood of x. Prove that G is a dense open subset of X. (Hint: The Baire category theorem may help.)

[17] **6.** Let H be a Hilbert space and let $A, B\colon H \to H$ such that for all $x, y \in H$,

$$\langle x \,|\, Ay \rangle = \langle Bx \,|\, y \rangle .$$

Prove that A and B are linear and continuous.