Analysis Qualifying Examination 2 **August 17, 2021**

Please start each problem on a new page and remember to write your code on each page of your answers.

You should exercise good judgement in deciding what constitutes an adequate solution. In particular, you should not try to solve a problem by just quoting a theorem that reduces what you are asked to prove to a triviality. If you are not sure whether you may use a particular theorem, ask the proctor.

1. Decide if the following statements are true or false. Prove your answers.

[4] **(a)** If $f \in L^\infty(\mathbf{R})$, then $f \in L^p(\mathbf{R})$ for all $p > 0$ and $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

[16] **(b)** If $f \in L^\infty([0,1])$, then $f \in L^p([0,1])$ for all $p > 0$ and $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

[16] **2.** Let $(X, \mathscr{A}, \mu)$ be a measure space and let $0 < p < 1$. Even though p is not greater than or equal to 1, one still defines $\|f\|_p$ to be $\left(\int |f|^p \, d\mu\right)^{1/p}$ for each measurable function $f \colon X \to \mathbf{C}$. However the usual Minkowski inequality does not hold. Instead what one might call a reverse Minkowski inequality holds. This means that

$$\|f + g\|_p \geq \|f\|_p + \|g\|_p$$

for all measurable functions $f, g \colon X \to [0, \infty)$. Prove this reverse Minkowski inequality.

[16] **3.** For each function $f \colon \mathbf{R} \to \mathbf{C}$ and for each $y \in \mathbf{R}$, define f^y on $\mathbf{R}$ by

$$f^y(x) = f(x - y).$$

Show that if $f \in L^p(\mathbf{R})$, where $p \in [1, \infty)$, then $\|f^y - f\|_{L^p(\mathbf{R})} \to 0$ as $y \to 0$.

[16] **4.** Let (f_n) be a sequence in ℓ^1 which converges weakly to 0. (This means that for each continuous linear functional H on ℓ^1, we have $H(f_n) \to 0$.) Prove that $\|f_n\|_1 \to 0$. (Note: This only applies to sequences, not to nets, so it does not mean that the weak and norm topologies on ℓ^1 are the same.)

[16] **5.** Let $f \colon \mathbf{R} \to \mathbf{C}$ be 1-periodic and continuously differentiable. For each $k \in \mathbf{Z}$, let

$$c_k = \int_0^1 e^{-2\pi i k x} f(x) \, dx.$$

Prove that $\sum_{k \in \mathbf{Z}} |c_k| < \infty$.

[16] **6.** Let X be a locally compact Hausdorff space. Let $C_0(X)$ be the vector space of continuous real-valued functions on X which tend to zero at infinity. As usual, give $C_0(X)$ the uniform norm, defined by

$$\|f\|_u = \sup \{ |f(x)| : x \in X \}$$

for each $f \in C_0(X)$. Let L be a linear functional on $C_0(X)$ such that for each $f \in C_0(X)$, if $f \geq 0$, then $L(f) \geq 0$. Prove that L is continuous.