What is Perron-Frobenius Theorem?

Sachin Gautam (June 28, 2022)

§0. Notations from linear algebra. Let $C \in M_{n\times n}(\mathbb{R})$. We say C is non-negative (resp. positive) - written as $C \geq 0$ (resp. $C > 0$) if $c_{k\ell} \geq 0$ (resp. $c_{k\ell} > 0$) for every $k, \ell \in \{1, \dots, n\}$. (Same for $\underline{u} \in \mathbb{R}^n$).

- $\chi_C(t) := \det(t \cdot I_n - C)$ is the characteristic polynomial of C.
- $\sigma(C) \subset \mathbb{C}$ set of eigenvalues of C - called spectrum of C. (root of $\chi_C(t)$)
- $\rho(C) = \mathrm{Max}\{|\lambda| : \lambda \in \sigma(C)\}$ - spectral radius of C.
- $\mathrm{Adj}(C)$ = adjoint matrix of C : [(k,ℓ)-th entry of $\mathrm{Adj}(C)$ is $(-1)^{k+\ell}$ times determinant of $(n-1)\times(n-1)$ size matrix obtained from C by deleting ℓ-th row and k-th column.]

Recall that $\boxed{C \cdot \mathrm{Adj}(C) = \mathrm{Adj}(C) \cdot C = \det(C) \cdot I_n}$

- For $C, D \in M_{n\times n}(\mathbb{R})$; $\boxed{C \underset{sym}{\sim} D}$ means that there is a permutation $\pi \in S_n$ such that $c_{k\ell} = d_{\pi(k),\pi(\ell)}$; $\forall\, k, \ell \in \{1, \dots, n\}$.

Definition : We say C is reducible if $C \underset{sym}{\sim} \begin{array}{c} \left[\begin{array}{c|c} C_1 & 0 \\ \hline C_3 & C_2 \end{array}\right] \begin{array}{l} \}p \\ \}n-p \end{array} \\ \quad\;\; \underbrace{\;\;}_{p} \;\; \underbrace{\;\;}_{n-p} \end{array}$.

Otherwise, C is said to be irreducible ('unzerlegbar' - the word Frobenius used - translates to "cannot be decomposed").

§1. Perron-Frobenius Theorem: Let $A \geq 0$ be an $n \times n$, irreducible matrix. Then $\rho = \rho(A) \in \mathbb{R}_{>0}$ is a simple eigenvalue of A, i.e., $\chi_A'(\rho) \neq 0$. Any (non-zero) eigenvector $\underline{u} \in \mathbb{R}^n$ with eigenvalue ρ (i.e., $A\underline{u} = \rho\underline{u}$) is either $\underline{u} > 0$ or $-\underline{u} > 0$ "sign-coherent".

Moreover, $\chi_A'(\rho) > 0$ and $B(t) := \mathrm{Adj}(t \cdot I_n - A) > 0 \;\; \forall t \geq \rho$.

Second part: $[A \in M_{n \times n}(\mathbb{R}_{\geq 0})$ irreducible$]$. Let us define

$$h := \#\{\lambda \in \sigma(A) : |\lambda| = \rho\}$$ (called index of imprimitivity of A)

Then: (i) $\sigma(A)$ is invariant under rotation by $2\pi/h$.

That is, $e^{2\pi i/h}\sigma(A) = \sigma(A)$.

(ii) $$\{\lambda \in \sigma(A) : |\lambda| = \rho\} = \{\rho \cdot e^{2\pi i k/h} : 0 \leq k \leq h-1\}.$$

(iii) $$A \underset{\text{sym}}{\sim} \begin{bmatrix} 0 & A_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & A_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & A_{h-1} \\ A_h & 0 & 0 & & \cdots & 0 & 0 \end{bmatrix}$$

"block cyclic form"

diagonal 0's are square blocks.

Examples: $A = \left[\begin{array}{cc|c} 2 & 0 & 0 \\ 1 & 2 & 0 \\ \hline 0 & 1 & 2 \end{array}\right]$ is reducible; $\chi_A(t) = (t-2)^3$

(Jordan block)

$\rho(A) = 2 \in \sigma(A) = \{2\}$; but the eigenvalue is not simple.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$ (cyclic matrix) is irreducible. $\chi_A(t) = t^3 - 1 = (t-1)(t - e^{2\pi i/3})(t - e^{4\pi i/3})$

$$\sigma(A) = \{1, e^{2\pi i/3}, e^{4\pi i/3}\} \; ; \quad \rho(A) = 1 \in \sigma(A).$$

§2. Some remarks. - (1) The conclusion $\rho(A) \in \sigma(A)$ is true for any $A \geq 0$ (not necessarily irreducible). However, it need not be a simple eigenvalue, as the example above shows.

(2) Oskar Perron (1880-1975) proved this theorem for $A > 0$ in 1907.

Georg Frobenius (1849-1917) generalized this result in 1912.

Andrey Markov (1856-1922) (independently) obtained similar statement for "stochastic matrices" in 1908.

Maurice Potron (1872-1942) (re-)introduced positive matrices - for models of "economic stability".

(3) Applications - in almost all of the applications of Perron-Frobenius theorem, one encounters the following "discrete evolutionary process"

Given $\boxed{\begin{array}{l} A \in M_{n\times n}(\mathbb{R}) \\ \underline{v} \in \mathbb{R}^n \end{array}}$ define $\underline{v}^{(0)} = \underline{v}$; $\underline{v}^{(1)} = A v^{(0)}$; $\underline{v}^{(2)} = A^2 v^{(0)} = A v^{(1)}$;

$$\ldots \quad v^{(N)} = A v^{(N-1)} = A^N v^{(0)} \quad \ldots$$

The theorem is used to address the questions: does $\lim_{N\to\infty} v^{(N)}$ exist? is it independent of the initial condition $\underline{v}$?

§3. Irreducibility condition. – Let $A \geq 0$ be an $n \times n$ matrix.

Note: $A > 0 \iff \forall\ \underline{v} \in \mathbb{R}^n_{\geq 0},\ \underline{v} \neq \underline{0}$, we have $A\underline{v} > 0$.

Lemma. – A is irreducible $\iff (I+A)^{n-1} > 0$.

Proof. – If A is reducible, both A and $I+A$ have block triangular form:

$$I + A \underset{\text{sym}}{\sim} \left[\begin{array}{c|c} C_1 & 0 \\ \hline C_3 & C_2 \end{array}\right] \Rightarrow (I+A)^N \underset{\text{sym}}{\sim} \left[\begin{array}{c|c} C_1^N & 0 \\ \hline * & C_2^N \end{array}\right] \not> 0.$$

Assume A is irreducible. Let $\underline{v} \in \mathbb{R}^n_{\geq 0} \setminus \{\underline{0}\}$. Let $\underline{w} = (I+A)\underline{v} = \underline{v} + A\underline{v}$.

Since $A \geq 0$, $\underline{v} \geq 0$, we get that $v_j > 0 \Rightarrow w_j > 0$.

Claim: $\underline{w}$ has strictly less 0's than $\underline{v}$. (Hence, $(I+A)^{n-1}\underline{v} > 0$).

If not, upon relabelling, $\underline{v} = \left[\begin{array}{c} \underline{0} \\ \hline \underline{v}' \end{array}\right] \begin{array}{l} \}p \\ \}n-p \end{array}$; $\underline{w} = \left[\begin{array}{c} \underline{0} \\ \hline \underline{w}' \end{array}\right] \begin{array}{l} \}p \\ \}n-p \end{array}$.

$(\underline{v}', \underline{w}' > 0)$.

Write $A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \begin{array}{l} \}p \\ \}n-p \end{array}$, with the column blocks of widths $\underbrace{\ }_{p}$, $\underbrace{\ }_{n-p}$. We get $A_{12}\underline{v}' = \underline{0}$. Since $\underline{v}' > 0$, this means $A_{12} = 0$ – contradicts irred. $\square$

Visualizing "irreducibility": $(I+A)^{n-1} = \sum_{p=0}^{n-1} \binom{n-1}{p} A^p$. Its

(k,l)-th entry is: $\delta_{kl} + \sum_{p=1}^{n-1} \binom{n-1}{p} \left(\sum_{j_1,\dots,j_{p-1}} a_{kj_1} a_{j_1 j_2} \cdots a_{j_{p-1} l} \right) > 0$

$\Rightarrow \forall\ k \neq l,\ \exists\ j_1, \dots, j_{p-1} \in \{1, \dots, n\}$ s.t. $a_{kj_1} a_{j_1 j_2} \cdots a_{j_{p-1} l} > 0$
$(1 \leq p \leq n-1)$

Let $\mathcal{G}_A$ be a directed graph on vertices $1, \dots, n$. For $k, l \in \{1, \dots, n\}$, $k \neq l$; we draw an arrow $k \to l$ if and only if $a_{kl} > 0$.

Then $\boxed{A \text{ is irreducible} \iff \mathcal{G}_A \text{ is connected}}$

Examples: $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \rightsquigarrow$
$$\begin{tikzcd} 1 & 2 \arrow[l] \\ 3 \arrow[ur] & \end{tikzcd}$$
cannot reach 2 or 3 from 1 – reducible.

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \rightsquigarrow$
$$\begin{tikzcd} 1 \arrow[rr] & & 2 \arrow[dl] \\ & 3 \arrow[ul] & \end{tikzcd}$$
connected. – hence the matrix is irred.

§4. Proof of the first part of Perron-Frobenius theorem. (Frobenius' Max-Min approach)

(4.1) Define $c : \mathbb{R}^n_{\geq 0} \setminus \{\underline{0}\} \longrightarrow \mathbb{R}_{\geq 0}$ by $\boxed{c(\underline{v}) = \underset{k : v_k \neq 0}{\text{Min}} \frac{(A\underline{v})_k}{v_k}}$.

That is, $\boxed{\begin{array}{l} c(\underline{v})\, \underline{v} \leq A\underline{v} \text{ and} \\ a\, \underline{v} \leq A\underline{v} \Rightarrow a \leq c(\underline{v}) \end{array}}$. Note: $c(\lambda \underline{v}) = c(\underline{v}) \ \forall \lambda \in \mathbb{R}_{>0}$;

Hence, in maximizing $c(\underline{v})$, we may assume $\|\underline{v}\|^2 = \sum_{j=1}^n v_j^2 = 1$.

$K := \{\underline{v} \in \mathbb{R}^n_{\geq 0} : \|\underline{v}\|^2 = 1\}$. (Issue: c is not continuous on K).

Replace K by $L = \{(I+A)^{n-1}\underline{v} : \underline{v} \in K\}$. Since $(I+A)^{n-1} > 0$, every $\underline{w} \in L$ is positive - hence c is continuous on L.

Note that $\begin{array}{l} \underline{v} \in K \\ \underline{w} = (I+A)^{n-1}\underline{v} \end{array} \Rightarrow c(\underline{v}) \leq c(\underline{w})$ [since $c(\underline{v})\,\underline{v} \leq A\underline{v}$
$\Rightarrow c(\underline{v})\,(I+A)^{n-1}\underline{v} \leq A\,(I+A)^{n-1}\underline{v}$
i.e. $c(\underline{v})\,\underline{w} \leq A\underline{w}$.]

So, in maximizing c, we can restrict our domain to L.

Now $c : L \to \mathbb{R}_{\geq 0}$ is continuous, and $L = (I+A)^{n-1}(K)$ is compact. Hence c attains its maximum value, say $r \in \mathbb{R}_{\geq 0}$ on some $\underline{w} \in L$.

$$\boxed{r = \operatorname{Max}\{c(\underline{u}) : \underline{u} \in L\} = c(\underline{w}) \text{ for some } \underline{w} \in L}.$$

Note that $c\left(\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}\right) = \operatorname{Min}_k \sum_\ell a_{k\ell} > 0$ by irreducibility of A

$\Rightarrow r > 0$.

(4.2) If $\underline{v} \in \mathbb{R}^n_{\geq 0} \setminus \{\underline{0}\}$ is such that $c(\underline{v}) = r$, then $\boxed{A\underline{v} = r\underline{v} \text{ and } \underline{v} > 0.}$

Proof - We know $c(\underline{v})\,\underline{v} \leq A\underline{v}$, i.e. $A\underline{v} - r\underline{v} \geq 0$. If not equal to $\underline{0}$; Apply $(I+A)^{n-1} > 0$ to get $A\underline{w} - r\underline{w} > 0$ $\quad (\underline{w} = (I+A)^{n-1}\underline{v})$.

$\Rightarrow \exists\ \varepsilon > 0$ small enough so that $A\underline{w} > (r+\varepsilon)\,\underline{w}$

$\Rightarrow$ $c(\underline{w}) \geq r + \varepsilon > r$ contradicting the definition of r.

Hence $A\underline{v} = r\underline{v}$ (i.e., $r \in \sigma(A)$). Moreover $\underline{w} = (I+A)^{n-1}\underline{v} = (1+r)^{n-1}\underline{v} > 0$

$\Rightarrow \underline{v} > 0$. $\square$

(4.3) $\boxed{r = \rho(A)}$ and $\boxed{\dim \operatorname{Ker}(r.I_n - A) = 1.}$.

Proof - Let $\underline{y} \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$ $(\alpha \in \sigma(A))$ be such that $A\underline{y} = \alpha\underline{y}$.

i.e. $\forall k$, $\sum_{\ell=1}^n a_{k\ell}\, y_\ell = \alpha \cdot y_k$. By triangle inequality, we get

$\sum_{\ell=1}^n a_{k\ell}\, |y_\ell| \geq |\alpha| \cdot |y_k|$. Hence, $A\underline{y}^+ \geq |\alpha| \cdot \underline{y}^+$ $\left(\underline{y}^+ = \begin{bmatrix} |y_1| \\ \vdots \\ |y_n| \end{bmatrix} \in \mathbb{R}^n_{\geq 0}\right)$

$\Rightarrow |\alpha| \leq c(\underline{y}^+) \leq r$.

As $r \in \sigma(A)$, we get $r = \rho(A)$.

$$\left[\begin{array}{l} \text{If } \alpha = r \text{ in the previous paragraph, then } \underline{y} \in \mathbb{R}^n \text{ and} \\ c(\underline{y}^+) = r \Rightarrow \underline{y}^+ > 0. \text{ Meaning } y_k \neq 0 \ \forall k. \end{array}\right] - (*)$$

Thus, if $\dim(\mathrm{Ker}(r.I_n - A)) > 1$, say $\underline{u}, \underline{v}$ are two linearly independent vectors in $\mathbb{R}^n$ with $\begin{array}{l} A\underline{u} = r\underline{u} \\ A\underline{v} = r\underline{v} \end{array}$, by taking their linear comb^n, we can get $\underline{y} \in \mathbb{R}^n$, $\underline{y} \neq \underline{0}$ but $y_k = 0$ for some k, contradicting $(*)$.

$\square$

(4.4) $\boxed{\chi_A'(\rho) > 0}$ and $\boxed{B(t) := \mathrm{Adj}(tI_n - A) > 0}$ $\forall t \geq \rho$ $(\rho = \rho(A) = r)$.

Proof. - Since $\dim(\mathrm{Ker}(\rho I_n - A)) = 1$, we get $\mathrm{rank}(\rho I_n - A) = n-1$.

$\Rightarrow$ some $(n-1)\times(n-1)$ minor of $\rho \cdot I_n - A$ must be non-zero, i.e. $B(\rho) \neq 0$.

Now $(\rho \cdot I_n - A)\, B(\rho) = \chi_A(\rho) = 0$ $\Rightarrow$ columns of $B(\rho)$ are eigenvectors for A with eigenvalue $= \rho$. Hence every non-zero column of $B(\rho)$ is either >0 or <0. Same is true for rows of $B(\rho)$ - by applying the previously proved results for A^T. Hence, either $B(\rho) > 0$ or $-B(\rho) > 0$.

In either case $\quad \chi_A'(\rho) = \sum_{k=1}^{n} B_{kk}(\rho) \neq 0.$

But ρ is the largest zero of $\chi_A(t)$ and $\underset{(=t^n+\cdots)}{\chi_A(t)} \to \infty$ as $t \to \infty$

$\Rightarrow \chi_A'(\rho) > 0$ (hence $\chi_A(t) > 0$ $\forall t > \rho$)

and $B(\rho) > 0$.

Finally $\quad B(t) = \chi_A(t) \cdot (t - A)^{-1} = \chi_A(t) \cdot \sum_{\ell=0}^{\infty} \frac{A^\ell}{t^{\ell+1}} > 0 \quad \forall t \geq \rho.$

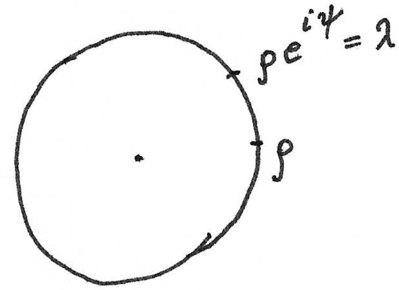
$\square$

§5. Proof of the second part of Perron-Frobenius theorem.

$A \geq 0$: irreducible $n \times n$ matrix. $\rho = \rho(A) \in \mathbb{R}_{>0}$. Let us fix $\underline{u} > 0$ so that $\boxed{A\underline{u} = \rho\,\underline{u}.}$

Let $h = \#\{\lambda \in \sigma(A) : |\lambda| = \rho\}$.

(5.1) There exists a diagonal matrix $D = \text{diagonal}(\eta_1, \dots, \eta_n)$ such that $|\eta_j| = 1$ (infact $D^h = I_n$; so $\eta_1, \dots, \eta_n$ are h-th roots of 1) and $\boxed{A = e^{2\pi i/h} \cdot DAD^{-1}}$.



Proof. Assume $h > 1$ (assertion is trivial for $h = 1$).

Choose $\lambda = \rho e^{i\psi} \in \sigma(A)$ with smallest argument $\psi \in (0, 2\pi)$.

Let $\underline{v} \in \mathbb{C}^n$ be an eigenvector $A\underline{v} = \lambda\underline{v}$. Again, by triangle ineq.*

$A\underline{v}^+ \geq |\lambda| \cdot \underline{v}^+ = \rho\underline{v}^+ \underset{(4.2)}{\Rightarrow} \underline{v}^+ \in \mathbb{R}_{>0}\,\underline{u}$. By scaling, if necessary, we may assume that $\underline{v}^+ = \underline{u}$; i.e. $\underline{v} = D\underline{u}$ where D is diagonal and $D^+ = I_n$. Now $A\underline{v} = \rho e^{i\psi}\underline{v}$ becomes $\underbrace{(e^{-i\psi} D^{-1} A D)}_{C}\,\underline{u} = \rho\,\underline{u}$.

So, $C^+ = A$ and $\boxed{C\underline{u} = \rho\underline{u}}$ ← take Real part to get

$$\left.\begin{array}{l} \mathrm{Re}(C)\,\underline{u} = \rho\,\underline{u} \\ C^+\,\underline{u} = \rho\,\underline{u} \end{array}\right\} \Rightarrow (C^+ - \mathrm{Re}(C))\,\underline{u} = 0.$$

But $C^+ - \mathrm{Re}(C) \geq 0$ and $\underline{u} > 0$. Hence $C^+ = \mathrm{Re}(C) = A \Rightarrow C = A$.

---

* recall the notation $\underline{v}^+_k = |v_k| \in \mathbb{R}_{\geq 0}$. Same for matrices $C^+_{k\ell} = |c_{k\ell}|$

That is, $A = e^{i\psi} D A D^{-1}$. As $\sigma(DAD^{-1}) = \sigma(A)$, we get $\sigma(A) = e^{i\psi}\sigma(A)$. Meaning $\sigma(A)$ is stable under rotations by ψ. Since there are exactly h eigenvalues on $\{|z| = \rho\}$ - we conclude that $\psi = \frac{2\pi}{h}$. Finally from $A = e^{2\pi i/h} D A D^{-1}$ the corresponding eigenvectors are $\underline{u}, D\underline{u}, \dots, D^{h-1}\underline{u}$ and $D^h \underline{u} = \underline{u} \Rightarrow D^h = I_n$. $\square$

(5.2) <u>Cyclic block decomposition</u>

$$A \underset{\text{sym}}{\sim} \begin{bmatrix} 0 & A_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & A_2 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & \ddots & & \\ 0 & 0 & 0 & 0 & \cdots & A_{h-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & A_{h-1} \\ A_h & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Scale to make sure 1 is an entry of D.

Permuting indices, if necessary, we may assume

$$D = \text{diagonal}\Big(\underbrace{1, \dots, 1}_{p_0}, \underbrace{\eta_1, \dots, \eta_1}_{p_1}, \dots, \underbrace{\eta_{s-1}, \dots, \eta_{s-1}}_{p_{s-1}}\Big)$$

$\eta_j = \exp\left(\frac{2\pi n_j}{h} \cdot i\right)$ $\quad 0 = n_0 < n_1 < \dots < n_{s-1} \leq h-1$. With these permuted indices, $A = (A_{kl})_{0 \leq k,l \leq s-1}$ $\quad A_{kl}$ is $p_k \times p_l$ size.

$\varepsilon DAD^{-1} = \varepsilon(\eta_k \eta_l^{-1} A_{kl}) = (A_{kl})$. $\quad [\varepsilon = e^{2\pi i/h}]$

$\Rightarrow A_{00} = A_{11} = \dots = 0$ and (taking $k=0$) - either all $A_{0l} = 0$ (contradicting irred.) or $n_1 = 1$ and $A_{02} = \dots = A_{0,s-1} = 0$.

Continuing this way, we get $n_1 = 1$, $n_2 = 2$, $\dots$, $n_{s-1} = h-1$ $(s = h)$ and hence A must have the form as claimed. $\square$

References. -

1. F. Gantmacher, Applications of the theory of matrices. Interscience Publishers.

2. T. Hawkins, Continued fractions and the origins of Perron-Frobenius theorem. Arch. Hist. Exact Sci (2008).

3. S. Karlin, Positive operators, J. Math. Mech (1959).

4. C. MacCluer, The many proofs and applications of Perron's theorem SIAM reviews. (2000)