ERDŐS DISCREPANCY PROBLEM

1. Statements and some examples

Conjecture 1 (Erdős 1932). *For any $f : \mathbb{N} \to \{\pm 1\}$, the discrepancy*

$$D(f) := \sup_{n,d \in \mathbb{N}} |f(d) + f(2d) + \ldots + f(nd)| = \infty.$$

Theorem 2 (Tao [13] 2015). *Let H be a complex Hilbert space and let $f : \mathbb{N} \to H$ be a function such that $\|f(n)\|_H = 1$ for all n. Then $D(f) := \sup_{n,d}\|f(d) + \ldots + f(nd)\|_H = \infty$.*

- If $H = \mathbb{C}$, then f takes values in unit circle.
- Quanta magazine [4]: "A magical answer to an Erdos 80 year old puzzle"
- Let f_N denote f restricted to $[N[= \{1, \ldots, N\}$. What is the largest N such that there exists $f : \mathbb{N} \to \{\pm 1\}$ for which $D(f_N) = 1$? $\max(\mathrm{N}) = 11$.
- What is the largest N for $D(f_N) = 2$? $\max(\mathrm{N}) = 1160$ using computer (Konev-Lisitsa [5] 2014). The solution is equal to the size of Wikipedia at that time.
- They also showed $max(N) \geq 130000$ for $D(f_N) = 3$.
- One may weaken the problem to allow all arithmetic progressions $a + jd$: Is it true that for all $f : \mathbb{N} \to \{-1, 1\}$,

$$\sup_{a,n,d \in \mathbb{N}} \left| \sum_{j=1}^{n} f(a + jd) \right| = \infty?$$

This is easily seen to be true thanks to van der Waerden theorem on two colors.

Furthermore, Roth [10] showed that if A is a subset of integers upto N with $|A| = \rho N$, then there exists an absolute positive constant c such that

$$\sup_{\substack{d \leq \sqrt{N} \\ a \bmod k}} \left| \sum_{\substack{n \in A \\ n \equiv a \bmod k}} 1 - \frac{\rho N}{q} \right| \geq c\sqrt{\rho(1-\rho)}N^{1/4}$$

In other words, the "new" discrepancy of an arbitrary sequence $f : \mathbb{N} \to \{-1, 1\}$ grows at the order of at least $N^{1/4}$.

Definition 3. $f : \mathbb{N} \to \mathbb{C}$ is *multiplicative* if $f(mn) = f(m)f(n) \forall (m, n) = 1$. f is *completely multiplicative* if $f(mn) = f(m)f(n) \forall m, n \in \mathbb{N}$.

- If f is completely multiplicative, then

$$|f(d) + f(2d) + \ldots + f(nd)| = |f(d)(f(1) + \ldots + f(n))|.$$

Then

$$D(f) = \sup_{N \in \mathbb{N}} |\sum_{n=1}^{N} f(n)|.$$

Corollary 4 (Tao [13] 2015)**.** *For every completely multiplicative function $f : \mathbb{N} \to S^1$,*

$$\sup_{N\in\mathbb{N}} |\sum_{n=1}^{N} f(n)| = \infty.$$

- Example: Liouville function $\lambda(n) = (-1)^{\text{number of prime factors of } n \text{ counted with multiplicity}}$ is completely multiplicative.
- Davenport: $L(N) := \sum_{n=1}^{N} \lambda(n) = O(\frac{N}{\log^A N})$ for any $A > 0$.
- Riemann hypothesis $L(N) = O(N^{1/2+\epsilon})$ for any $\epsilon > 0$.
- Pólya [9] conjectured $L(N) \leq 0$ for all $N \geq 2$. It was disproved by Haselgrove [2] in 1958 in which he estimated a counterexample $N \approx 1.845 \times 10^{361}$. The smallest counterexample known so far was provided by Tanaka [11] in 1980 with $N = 906150257$.
- Borwein-Ferguson-Mossinghoff [1] 2008: There exist infinitely many N such that $\sum_{n=1}^{N} \lambda(n) > 0.00618\sqrt{N}$.
- Humphries [3] 2011: There exist infinitely many N such that $\sum_{n=1}^{N} \lambda(n) < -1.389\sqrt{N}$.
- Central limit theorem implies $D(f_N)$ grows like $\sqrt{N}$ for a random sequence f.

2. History

- 1932: conjectured by Erdos
- Ben Green: This problem is one of Erdos' favorite. He mentioned it many times over the years, particularly towards the end of his life.
- Andrew Granville: Everyone in the subject has whetted their teeth on this and failed. It's one of those problems that nobody has really written a sensible paper about, because no one had a clever idea.
- Late 2009: Gowers polled for next Polymath project
- Jan 2010: Gowers "emergency post".

  He just came back from Egypt, he wrote an emergency post: "This is an emergency post, since the number of comments on the previous post about Erdos discrepancy problem has become unwieldy while I've been away enjoying a bit of sunshine in Luxor."

  "The results of the polls, as they then stood, about the next Polymath project were already suggesting that the Erdos discrepancy problem was the clear favourite"
- Polymath5 is active until 2012
- 2014: Konev-Lisitsa for $D(f) = 2$
- Jan 2015: Matomäki-Radziwiłł [6] proved "Multiplicative functions in short intervals":

Theorem 5 (M-R)**.** *Let $f : \mathbb{N} \to [-1, 1]$ be a multiplicative function. There exist $C, C' > 1$ such that for any $h \leq X$ and $\delta > 0$,*

$$\left|\mathbb{E}_{n\in[x,x+h]} f(n) - \mathbb{E}_{n\in[X,2X]} f(n)\right| \leq \delta + C' \frac{\log\log h}{\log h}$$

for all but a "small number" of $x \in [X, 2X]$ (more precisely, at most

$$CX \left(\frac{(\log h)^{1/3}}{\delta^2 h^{\delta/25}} + \frac{1}{\delta^2 (\log x)^{1/50}} \right)$$

integers $x \in [X, 2X]$.)

The remarkable of this theorem is that it shows Liouville function exhibits cancelation in almost all interval $[x, x+h]$ as long as $h \to \infty$ (however slowly, even in the case $h \ll x$). Before, it is only known for h grows like a power of x.

- March 2015: Matomäki-Radziwiłł-Tao [7] proved "Averaged form of Chowla's conjecture"

Theorem 6 (M-R-T)**.**

$$\mathbb{E}_{h_1,\dots,h_k\in[H]} \left|\mathbb{E}_{n\in[N]}\lambda(n+h_1)\cdots\lambda(n+h_k)\right| = o_{H\to\infty}(1) + o_{N\to\infty}(1).$$

- 4 Sep 2015: Matomäki-Radziwiłł-Tao [8] proved "Sign patterns of Liouville function"

Theorem 7 (M-R-T)**.** *Each of the 8 sign patterns in $\{-1,1\}^3$ is attained by λ for a set of positive lower density.*

- 6 Sep: Tao [12] wrote a blog post about this result. Mention Sudoku flavor.
- 9 Sep: Stroinski: Sudoku-flavor reminds him of Erdos discrepancy problem. Can the argument be applied to tackle the conjecture?
- Tao: No! Because the technique works for short interval where ($h \ll x$) while in discrepancy problem, both n and d are large.
- Tao [4]: In one afternoon, waiting for his son to get out of a piano lesson, the answer came to him.
- 17 Sep 2015: Tao proved Erdos discrepancy conjecture.
- 4 Nov 2015: Accepted in Discrete Analysis. First article in this new journal.
- 17 Sep 2015: Tao [14] proved logarithmic 2-point Chowla and Elliot's conjectures:

Theorem 8 (Tao [14])**.** *For multiplicative function f, if $\mathbb{E}^{\log}_{n\in[N]} f(n+h_1)\overline{f(n+h_2)}$ is large then f must correlates with Dirichlet characters (more precisely, pretend to be a twisted Dirichlet characters).*

3. Some almost counter-examples

- (Dirichlet character) $\chi_q : \mathbb{N} \to \mathbb{C}$ is a Dirichlet character of period q if
  - $\chi_q(n) = 0$ if $(n, q) > 1$
  - $\chi_q : (\mathbb{Z}/q\mathbb{Z})^\times \to S^1$ is a homomorphism
  - χ_q is periodic with period q.
- Principal Dirichlet character: $\chi_q(n) = 1$ for all $(n, q) = 1$.
- $q = 3$. Non principle Dirichlet character $\chi_3 : \mathbb{N} \to \{0, \pm 1\}$ with $\chi_3(n) \equiv n \bmod 3$.

$$\chi_3(n) = \begin{cases} 0 \text{ if } n \equiv 0 \bmod 3 \\ 1 \text{ if } n \equiv 1 \bmod 3 \\ -1 \text{ if } n \equiv -1 \bmod 3 \end{cases}$$

- Every non-principal character χ_q has sum zero in every interval of length q. Thus, $D(\chi_q) \leq q$.

- (Borwein-Choi-Coons example). Define completely multiplicative function $\tilde{\chi}_3 : \mathbb{N} \to \{\pm 1\}$ by setting
$$\tilde{\chi}_3(p) = \begin{cases} 1 \text{ if } p = 3 \\ \chi(p) \text{ if } p > 3. \end{cases}$$
Then $\sum_{n=1}^{N} \tilde{\chi}_3(n) =$ number of 1s in base 3 expansion of N. Thus $D(\tilde{\chi}_{3,N}) \gg \log N$.
- Cesàro smooth discrepancy is bounded
$$\sup_N |\sum_{n=1}^{N}(1 - \frac{n}{N})\tilde{\chi}_3(n)| < \infty.$$

4. Ideas of proof

- Assume the conjecture is false. Then there is a completely multiplicative function has bounded partial sum. (Polymath)
- Use van der Corput to show completely multiplicative function with bounded partial sum must correlates with itself, i.e.
$$\mathbb{E}_{n \in [N]} f(n + h_1)\overline{f(n + h_2)}$$
is large.
- If $\mathbb{E}^{\log}_{n \in [N]} f(n + h_1)\overline{f(n + h_2)}$ is large then f must correlates with characters. (Tao's 2 point Elliot)
- If a completely multiplcative function correlates with Dirichlet character, then discrepancy goes to infinity. (Polymath)

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