What is a generalized Bernoulli number?

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Abstract

After a review of the classical Bernoulli numbers, we explore how Dirichlet L-functions lead to the generalized Bernoulli numbers introduced by Leopoldt in 1958.

1 Bernoulli numbers

The Bernoulli numbers appear in an enormous number of different parts of mathematics, and consequently can be introduced and motivated in many ways. We will introduce them as they first arose in history: sums of powers.

1.1 History of computing sums of powers

Calculating sums of the form $1^n + 2^n + \cdots + k^n$ is a basic and ancient problem in number theory. The formulas for sums of nth powers, up to $n = 4$, were known in antiquity; the following are, as far as I can tell, the earliest attested appearances of them:

- The formula for triangular numbers was known to the Pythagoreans (600–400 BC):
$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$
- The formula for pyramidal numbers appears in the work of Archimedes, c. 225 BC:
$$1^2 + 2^2 + \cdots + k^2 = \frac{2k^3 + 3k^2 + k}{6}$$
- The following identity for sums of cubes was discovered by Nicomachus (c. 60–c. 120 AD):
$$1^3 + 2^3 + \cdots + k^3 = (1 + 2 + \cdots + k)^2$$
- The formula for sums of fourth powers was derived by Ibn al-Haytham (c. 965–c. 1040, born in modern-day Iraq):
$$1^4 + 2^4 + \cdots + k^4 = \frac{6k^5 + 15k^4 + 10k^3 - k}{30}$$

Next, we turn to the more interesting question of finding a uniform method to derive such formulas. Thomas Harriot (c. 1560–1621, England) derived the four formulas above by iteratively taking differences of the sequences of powers. Johann Faulhaber (1580–1635, Germany) had much more success and was able to compute the sums of nth powers up to $n = 17$. The more well-known mathematicians Fermat and Pascal also contributed to this problem; Fermat was interested in computing the integral of x^n, and Pascal introduced his eponymous triangle to attack the problem.

The problem of providing a general formula for all exponents was accomplished in two places independently, and nearly the same time but at opposite points on the globe. More famously, in 1713, Jakob Bernoulli's *Artis Conjectandi* was posthumously published, which contained a general formula for sums of powers and introduced the quantities now called Bernoulli numbers, which appear in the coefficients of Bernoulli's formula. However, a year earlier in 1712, a work of Seki Takakazu had been published posthumously, in which Bernoulli numbers are defined and a formula essentially identical to Bernoulli's for sums of powers is derived.

For a more detailed history, see [Bee10] or [Lar19, §§1–2].

1.2 A derivation of the Bernoulli–Seki formula

Let $S_n(k) \coloneqq \sum_{a=1}^k a^n$ for $k, n \geq 0$. We'll give a modern derivation, using exponential generating functions, of the formula for $S_n(k)$ obtained by Bernoulli and Seki, and define the Bernoulli numbers along the way.

We begin by forming the exponential generating function of $S_n(k)$ as n varies:

$$\sum_{n\geq 0} S_n(k)\frac{T^n}{n!} = \sum_{a=1}^{k}\sum_{n\geq 0}\frac{(aT)^n}{n!} = \sum_{a=1}^{k} e^{aT} = e^T\frac{e^{kT}-1}{e^T-1}.$$

We wish to expand the final expression as a formal power series in T which, by comparing coefficients on both sides, will gives us a formula for $S_n(k)$. To do this, we write

$$e^T\frac{e^{kT}-1}{e^T-1} = \frac{Te^T}{e^T-1}\cdot\frac{e^{kT}-1}{T} = \left(\sum_{m\geq 0} B_m\frac{T^m}{m!}\right)\left(\sum_{n\geq 0}\frac{k^{n+1}T^n}{(n+1)!}\right), \tag{1}$$

where we *define* the Bernoulli numbers B_m by the identity of formal power series $Te^T/(e^T-1) = \sum_{m\geq 0} B_m T^m/m!$. Equating the coefficient on T^n on both sides in (1) gives

$$\frac{S_n(k)}{n!} = \sum_{j=0}^{n}\frac{B_{n-j}}{(n-j)!}\cdot\frac{k^{j+1}}{(j+1)!},$$

and multiplying both sides by $n!$ and simplifying yields

$$S_n(k) = \frac{1}{n+1}\sum_{j=0}^{n}\binom{n+1}{j}B_{n-j}k^{j+1},$$

which is the desired expression for $S_n(k)$ as a polynomial in k.

1.3 Computation of the Bernoulli numbers

Our derivation of the Bernoulli–Seki formula immediately raises the problem of finding an effective way to compute the Bernoulli numbers. We begin by writing

$$e^T = \frac{e^T - 1}{T} \sum_{n \geq 0} B_n \frac{T^n}{n!} = \left(\sum_{m \geq 0} \frac{T^m}{(m+1)!} \right) \left(\sum_{n \geq 0} B_n \frac{T^n}{n!} \right).$$

Equating the coefficient of T^n on both sides gives the initial condition $B_0 = 1$ (which tells us that the polynomial appearing in the Bernoulli–Seki formula has degree exactly $n + 1$) and recurrence relation

$$\sum_{j=0}^{n} \frac{1}{(n-j+1)!} \cdot \frac{B_j}{j!} = \frac{1}{n!}.$$

This relation is simplified by multiplying by $n!$:

$$\sum_{j=0}^{n} \frac{\binom{n+1}{j}}{n+1} B_j = 1. \tag{2}$$

The coefficient on B_n here is 1, allowing for easy computation. For example,

$$B_1 = 1 - \frac{1}{2} B_0 = \frac{1}{2}, \quad B_2 = 1 - \frac{1}{3} B_0 - \frac{3}{3} B_1 = \frac{1}{6}.$$

Moreover, we see that $B_n = 0$ for $n \geq 3$ odd, since

$$\frac{Te^T}{e^T - 1} - \frac{T}{2} = \frac{T(e^T + 1)}{2(e^T - 1)}$$

is an even function of T.

It is worth mentioning that a large portion of the literature adheres to the convention $B_1 = -1/2$, defining the Bernoulli numbers instead by the generating function $T/(e^T - 1)$. However, we will see below that we really want $B_1 = +1/2$. The essay [Lus13] elaborates many more reasons to use the +-sign.

1.4 Addendum: congruences among Bernoulli numbers

Of the many interesting properties of Bernoulli numbers, one of the deepest and most important to number theory is the congruences they satisfy. Kummer in 1851 was the first to discover such a congruence, namely

$$\frac{B_k}{k} \equiv \frac{B_\ell}{\ell} \pmod{p}$$

whenever $k \equiv \ell \pmod{p-1}$ [Kum51]. He used this to show that the arithmetic properties of Bernoulli numbers are intertwined with the arithmetic of cyclotomic fields.

This connection allowed him, among other things, to prove Fermat's last theorem for a large number of exponents. Kubota and Leopoldt in 1964 used these congruences to define a p-adic analogue of Riemann's zeta-function [KL64], the key ingredient being the identity (4) below. See [IR90, Ch. 15] or [Was97] for much more details on congruences among Bernoulli numbers and how these quantities arise in the theory of cyclotomic fields.

2 Generalized Bernoulli numbers

2.1 Definition and main theorem

Here is another classical appearance of Bernoulli numbers. In 1734, Euler discovered that $\sum_{n\geq 1} n^{-2} = \pi^2/6$. He was able to show further that for any $k \geq 1$,

$$\zeta(2k) = \frac{(-1)^{k-1}(2\pi)^{2k}B_{2k}}{2(2k)!}.$$

Exercise 3. Use Euler's formula for $\zeta(2k)$ to compute the sign of B_{2k} and bound $|B_{2k}|$ from below.

In 1859, Riemann proved that ζ extends (uniquely) to a holomorphic function on $\mathbb{C} \setminus \{1\}$. To do so, he showed that ζ satisfies a certain functional equation relating $\zeta(1-s)$ to $\zeta(s)$, which allows one to express the equality above in the considerably simpler form

$$\zeta(1-k) = -\frac{B_k}{k}, \tag{4}$$

which is valid for all integers $k \geq 1$.

The zeta-function was generalized by Dirichlet in 1837 to prove that there are infinitely many primes in each residue class $a \pmod f$ with $(a, f) = 1$. His generalization is the following. First, a *Dirichlet character* is a group homomorphism $\chi\colon (\mathbb{Z}/f)^\times \to \mathbb{C}^\times$. The positive integer f is called the *conductor* of χ. We extend the domain of χ to all of $\mathbb{Z}$ by saying $\chi(a) = 0$ if $(a, f) > 1$. Finally, we define the *Dirichlet L-function associated to χ*

$$L(s,\chi) \coloneqq \sum_{n\geq 1} \frac{\chi(n)}{n^s},$$

which converges to a holomorphic function in the half-plane $\{\operatorname{re} s > 1\}$. Observe that $L(s, \chi_0) = \zeta(s)$, where χ_0 denotes the unique character of conductor 1. The Dirichlet L-functions share many properties with ζ; for example, they have Euler products, analytic continuations, functional equations, and so on. Of course, preceding Riemann, Dirichlet only considered $L(s,\chi)$ with $s \in \mathbb{R}$, and had no notion of analytic continuation or functional equation.

We finally define the generalized Bernoulli numbers. Namely, for χ a Dirichlet character, we define $B_{k,\chi}$ by generalizing (4) to Dirichlet L-functions:

$$L(1-k,\chi) =: -\frac{B_{k,\chi}}{k}.$$

By definition and (4), $B_{n,\chi_0} = B_n$ for all n. This, in light of our definition of the classical Bernoulli numbers, raises the following question: what is the exponential generating function of the $B_{n,\chi}$? The main theorem of this talk is the following:

Theorem 5. *Fix χ a Dirichlet character of conductor f.*

(*a*) *There is an analytic continuation of $L(s,\chi)$ to $\mathbb{C} \setminus \{1\}$ (so that the definition of $B_{n,\chi}$ makes sense).*

(*b*) *We have the identity of formal power series*

$$\sum_{n\geq 0} B_{n,\chi}\frac{T^n}{n!} = \sum_{a=1}^{f} \frac{\chi(a)Te^{aT}}{e^{fT}-1}.$$

Exercise 6. Use the exponential generating function to express $\sum_{a=1}^{fk}\chi(a)a^n$ as a polynomial in k of degree $\leq n+1$ whose coefficients are defined in terms of the $B_{n,\chi}$.

Exercise 7. Use the exponential generating function to obtain a recursive formula for $B_{n,\chi}$ along the lines of (2).

2.2 Proof of the main theorem

Our proof roughly follows [Iwa72, Appx.]. We'll encounter the auxiliary function

$$G(z) \coloneqq \sum_{a=1}^{f} \frac{\chi(a)e^{-az}}{1-e^{-fz}}.$$

This is holomorphic on $\mathbb{C} \setminus 2\pi i/f \cdot \mathbb{Z}$. Observe that $zG(z) = F(-z)$, where

$$F(z) \coloneqq \sum_{a=1}^{f} \frac{\chi(a)ze^{az}}{e^{fz}-1}$$

is the function whose nth Taylor coefficient we claim to be $B_{n,\chi}/n!$. Also, we have

$$G(z) = \sum_{n\geq 1}\chi(n)e^{nz}$$

(sum over each residue class modulo f individually).

We begin the proof, rather similarly to Riemann's proof of the functional equation for his zeta-function, with the following observation:

$$n^{-s}\Gamma(s) = n^{-s}\int_0^\infty e^u u^s \,\frac{du}{u} = n^{-s}\int_0^\infty e^{nt}(nt)^s\,\frac{dt}{t} = \int_0^\infty e^{nt}t^s\,\frac{dt}{t},$$

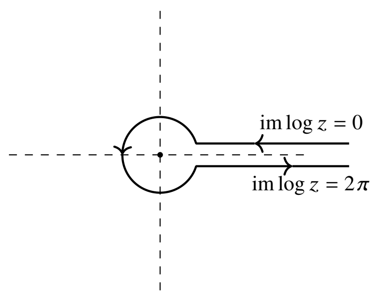
where the substitution is $t = nu$. Therefore (leaving, as we'll do throughout, issues of convergence in the reader's capable hands)

$$L(s,\chi)\Gamma(s) = \sum_{n\geq 1} \chi(n) n^{-s}\Gamma(s) = \sum_{n\geq 1}\chi(n)\int_0^\infty e^{nt} t^s \,\frac{dt}{t} = \int_0^\infty \Bigg(\sum_{n\geq 1}\chi(n)e^{nt}\Bigg) t^s \,\frac{dt}{t}$$
$$= \int_0^\infty G(t) t^s \,\frac{dt}{t}.$$

The idea is now to replace integration along $(0,\infty)$ with integration over a closely related contour to obtain an entire function, whose values we'll later be able to compute via the residue theorem. Fix $\varepsilon \in (0, 2\pi/f)$. Let γ denote the contour $(\infty,\varepsilon) + \{|z| = \varepsilon\} + (\varepsilon,\infty)$, and define

$$H(s) := \int_\gamma G(z) z^s \,\frac{dz}{z},$$

where $z^s := e^{s\log z}$ and $\operatorname{im}\log z$ varies continuously along γ from 0 to 2π.



(Notice here that we are treating z^s as a multi-valued function, so the integrals along the paths (∞,ε) and (ε,∞) do not cancel. To avoid this and make the situation more transparent, one can replace z by e^w in the definition of $H(s)$, and γ by the rectangular path $(\infty, \log\varepsilon) + [\log\varepsilon, \log\varepsilon + 2\pi i] + (\log\varepsilon + 2\pi i, \infty + 2\pi i)$.)

Since $G(z)$ is holomorphic inside γ except at 0 and decays exponentially as $\operatorname{re} z \to \infty$, the function H is entire and does not depend on ε. Since

$$H(s) = \int_\infty^\varepsilon G(t)t^s\,\frac{dt}{t} + \int_{|z|=\varepsilon} G(z)z^s\,\frac{dz}{z} + e^{2\pi i s}\int_\varepsilon^\infty G(t)t^s\,\frac{dt}{t}$$

and the middle integral vanishes for $\operatorname{re} s > 1$ as $\varepsilon \to 0$, the independence from ε allows us to take $\varepsilon \to 0$ and get

$$H(s) = (e^{2\pi i s} - 1)\int_0^\infty G(t)t^s\,\frac{dt}{t} = (e^{2\pi i s} - 1)L(s,\chi)\Gamma(s).$$

for $\operatorname{re} s > 1$. Thus we obtain

$$L(s,\chi) = \frac{H(s)}{(e^{2\pi i s} - 1)\Gamma(s)}$$

for re $s > 1$, and the right hand side is known to be holomorphic for $s \in \mathbb{C} \setminus \mathbb{Z}$, and in fact is holomorphic at nonpositive integers since Γ has (simple) poles at such values. Thus we have an analytic continuation of $L(s, \chi)$ to all of $\mathbb{C} \setminus 1$.

Now fix $k \geq 1$. Since $1 - k$ is a simple pole of Γ of residue $(-1)^{k-1}/(k-1)!$ (first compute $\lim_{s \to 0} s\Gamma(s)$, and then apply the functional equation of Γ), we have

$$\lim_{s \to 1-k} (e^{2\pi i s} - 1)\Gamma(s) = 2\pi i \frac{(-1)^{k-1}}{(k-1)!}.$$

On the other hand,

$$H(1-k) = \int_\gamma G(z) z^{1-k} \frac{dz}{z} = \int_{|z|=\varepsilon} G(z) z^{1-k} \frac{dz}{z},$$

the latter equality since k is an integer (so z^{1-k} is holomorphic on $\mathbb{C} \setminus \{0\}$). Thus

$$H(1-k) = 2\pi i \operatorname*{res}_{z=0} G(z) z^{-k} = 2\pi i \operatorname*{res}_{z=0} F(-z) z^{-1-k} = 2\pi i \frac{(-1)^k C_{k,\chi}}{k!},$$

where $C_{k,\chi}/n!$ is the nth Taylor coefficient of $F(z)$. Therefore

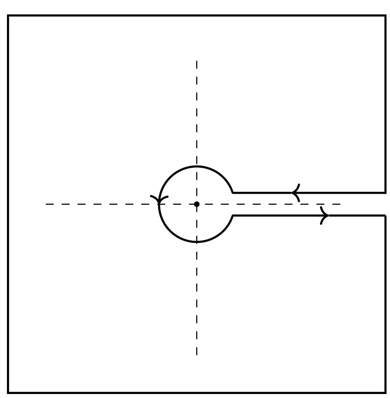
$$L(1-k, \chi) = \frac{2\pi i (-1)^k C_{k,\chi}/k!}{2\pi i (-1)^{k-1}/(k-1)!} = -\frac{C_{k,\chi}}{k},$$

so $C_{k,\chi} = B_{k,\chi}$, as desired. □

2.3 Remarks on the proof

In the previous proof we computed that $H(1) = cB_{0,\chi}$ with $c \neq 0$. It's easy to see from the generating function that $B_{0,\chi} = \sum_{a=1}^{f} \chi(a)$, which is 1 if χ is a trivial character (i.e. corresponding to a trivial homomorphism) and 0 otherwise. Consequently, $L(s, \chi)$ is holomorphic at $s = 1$ unless χ is trivial, and otherwise has a simple pole at $s = 1$.

Also, we are not too far from the functional equation for $L(s, \chi)$. To derive this, consider replacing, in the definition of $H(s)$, the contour γ with



This integral picks up the residues of $G(z)z^{s-1}$ away from 0, which can be readily computed. Moreover, if $\operatorname{re} s < 0$, then the integral along the outer square vanishes as the edge length grows, hence taking the limit returns us to $H(s)$. Comparing the resulting expression for $H(s)$ as a sum of residues with the expression $H(s) = (e^{2\pi i s} - 1)L(s, \chi)\Gamma(s)$ above yields the functional equation.

Exercise 8. Use the idea in the previous paragraph to prove the functional equation.

2.4 History of the generalized Bernoulli numbers

It seems that generalized Bernoulli numbers appear for the first time in a 1891 paper of Berger [Ber91, §II], who defines them only for those Dirichlet characters χ corresponding to a Kronecker symbol. Berger gives formulas for $L(2k \pm 1, \chi)$ in terms of $B_{2k\pm1,\chi}$. They later appear, again for a special class of Dirichlet characters, in a 1952 paper of Ankeny–Artin–Chowla [AAC52, Thm. 3] in connection to congruences satisfied by the class number of a real quadratic field. Finally, Leopoldt, in a 1958 paper [Leo58], defines them in general and studies them in detail. He observes the identity $L(1 - k, \chi) = -B_{k,\chi}/k$, that the sums $\sum \chi(a)a^n$ can be expressed using the $B_{k,\chi}$, and studies the arithmetic properties of the generalized Bernoulli numbers. In each of these papers, the $B_{n,\chi}$ are defined in terms of the exponential generating function.

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