What is...
Ptolemy's inequality in non-zero curvature?

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   - Ptolemy's inequality in spherical and hyperbolic geometry
3. $\mathbb{R}$-trees

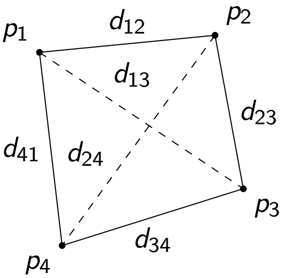
Ptolemy's inequality

Theorem

For any four points $p_1, p_2, p_3, p_4 \in \mathbb{R}^2$,

$$d_{13}d_{24} \leq d_{12}d_{34} + d_{23}d_{41},$$

with equality if, and only if, the points are collinear or concyclic.



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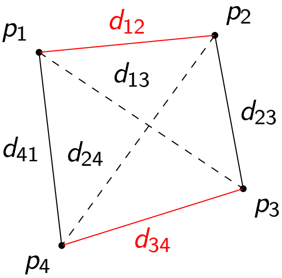

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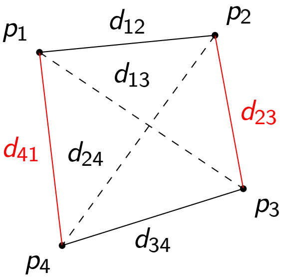

Claudius Ptolemy

- Born in Alexandria, Egypt (c. 100-170 AD).
- He was a mathematician, astronomer, geographer, and music theorist.

The Almagest

- It is considered the culmination of Greek astronomy [Ped11].
- It contains:
  - Table of chords,
  - The Ptolemaic model of the universe (geocentric),
  - Varied astronomical calculations.
- It influenced Western astronomy for over a millennium.
- Only Greco-Roman astronomical work that survives to this day.

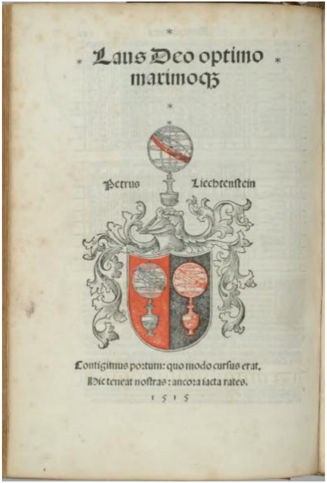

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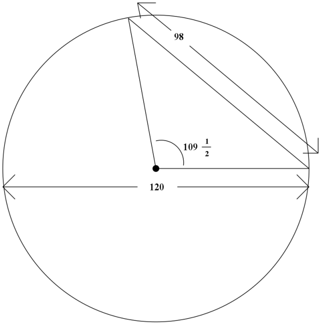
Table of chords



$$\text{ch}(\theta) = 2R\sin(\theta/2)$$

- The table recorded the length of chords in a circle where:
    - Diameter is 120
    - Subtended angle goes from 0° to 180° in increments of half a degree.

Calculation of chords

Tool	Angles	
Elementary polygons	$36°, 60°, 72°, 90°$	

1

2

Calculation of chords

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$\frac{\text{ch}(x)}{x} < \frac{\text{ch}(y)}{y}$ if $x > y$	$3/4° < 1° < 3/2°$	

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Calculation of chords

Tool	Angles	Known earlier? [Ped11]
Elementary polygons	$36°, 60°, 72°, 90°$	Early Greeks[1]
Pythagorean theorem	$108°, 120°, 144°$	Pythagoras (500 BC)
Difference of angles	$18°, 12°, 6°$	Archimedes (200s BC)
Half-angle identity	$3°, 3/2°$ and $3/4°$	Archimedes (200s BC)
$\frac{\text{ch}(x)}{x} < \frac{\text{ch}(y)}{y}$ if $x > y$	$3/4° < 1° < 3/2°$	Aristarchos (300s BC)[2]
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Calculation of chords

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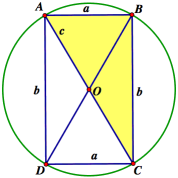
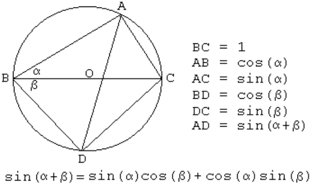
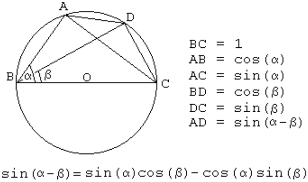
[1]Ptolemy computed the chords in sexagesimal numbers.

[2]Ptolemy realized that he could compute $\mathrm{ch}(1°)$ using this inequality.

Corollaries of Ptolemy's theorem







Images from:

- https://www.researchgate.net/publication/344787296_Ptolemy_Through_the_Centuries_for_Fullerton_Mathematical_Circle
- https://www.cut-the-knot.org/proofs/sine_cosine.shtml

Overview

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Cayley-Menger determinant

Let $X = \{p_1, p_2, \ldots, p_n\}$ be a finite metric space with distances $d_{ij} = d_X(p_i, p_j)$. Define

$$\mathrm{CM}(p_1, \ldots, p_n) = \left(\begin{array}{c|cccc} 0 & 1 & 1 & \cdots & 1 \\ \hline 1 & 0 & d_{12}^2 & \cdots & d_{1n}^2 \\ 1 & d_{21}^2 & 0 & \cdots & d_{2n}^2 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & d_{n1}^2 & d_{n2}^2 & \cdots & 0 \end{array}\right).$$

The quantity $\Delta(p_1, \ldots, p_n) = \det\big(\mathrm{CM}(p_1, \ldots, p_n)\big)$ is called the *Cayley-Menger determinant*.

Facts about the Cayley-Menger determinant

- Let S be the convex hull of the $n+1$ points $p_1, \ldots, p_{n+1} \in \mathbb{R}^n$. The n-dimensional volume of S is given by

$$\mathrm{Vol}_n^2(S) = \frac{(-1)^{n+1}}{2^n (n!)^2} \Delta(p_1, \ldots, p_{n+1}).$$

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- For $p_1, \ldots, p_{n+2} \in \mathbb{R}^n$, $\Delta(p_1, \ldots, p_{n+2}) = 0$.
- When $n = 2$, the Cayley-Menger determinant gives Heron's formula for the area of a triangle A with side lengths a, b, c [DS05]

$$\begin{aligned} 16A^2 &= -\Delta(p_1, p_2, p_3) \\ &= (a+b+c)(-a+b+c)(a-b+c)(a+b-c). \end{aligned}$$

Facts about the Cayley-Menger determinant

$$\mathrm{Vol}_n^2(S) = \frac{(-1)^{n+1}}{2^n(n!)^2}\Delta(p_1, \ldots, p_{n+1}).$$

- Given a finite metric space (X, d_X), there exists an isometric embedding $X \hookrightarrow \mathbb{R}^n$ if and only if there exist $p_1, \ldots, p_{n+1} \in X$ such that:
  - $(-1)^k \Delta(p_1, \ldots, p_k) \geq 0$, for every $k = 2, \ldots, n+1$,
  - $\Delta(p_1, \ldots, p_{n+1}, x) = \Delta(p_1, \ldots, p_{n+1}, y) = \Delta(p_1, \ldots, p_{n+1}, x, y) = 0$ for all $x, y \in X$.

  See [Blu70, Thm 42.2].

A related matrix

Let $X = \{p_1, p_2, \ldots, p_n\}$ be a finite metric space with distances $d_{ij} = d_X(p_i, p_j)$. Let

$$P(p_1, \ldots, p_n) = \begin{pmatrix} 0 & d_{12}^2 & \cdots & d_{1n}^2 \\ d_{21}^2 & 0 & \cdots & d_{2n}^2 \\ \vdots & \vdots & & \vdots \\ d_{n1}^2 & d_{n2}^2 & \cdots & 0 \end{pmatrix}.$$

Denote its determinant as $\gamma(p_1, \ldots, p_n) = \det\big(P(p_1, \ldots, p_n)\big)$.

Facts about this matrix

- The points $p_1, \ldots, p_{n+2} \in \mathbb{R}^n$ lie on an $(n-1)$-sphere or a $(n-1)$-hyperplane if, and only if,

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- If $n = 2$, $\gamma(p_1, p_2, p_3, p_4)$ factors as

$$\begin{aligned} \gamma(p_1, p_2, p_3, p_4) = -&(d_{13}d_{24} + d_{12}d_{34} + d_{23}d_{41}) \\ &\cdot (-d_{13}d_{24} + d_{12}d_{34} + d_{23}d_{41}) \\ &\cdot (+d_{13}d_{24} - d_{12}d_{34} + d_{23}d_{41}) \\ &\cdot (+d_{13}d_{24} + d_{12}d_{34} - d_{23}d_{41}). \end{aligned}$$

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Spherical and hyperbolic spaces

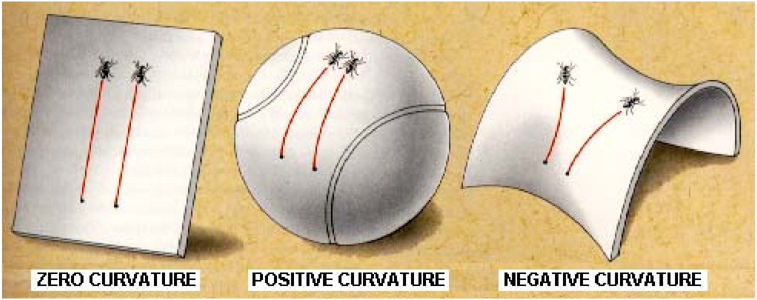

Figure: From "A Journey into Gravity and Spacetime" by J. A. Wheeler.

Spherical and hyperbolic spaces

Given $x, y \in \mathbb{R}^{d+1}$, let

$$\begin{aligned}\langle x, y\rangle &= x_0y_0 + x_1y_1 + \cdots + x_dy_d, \text{ and}\\ \langle x|y\rangle &= -x_0y_0 + x_1y_1 + \cdots + x_dy_d.\end{aligned}$$

Spherical and hyperbolic spaces

Given $x, y \in \mathbb{R}^{d+1}$, let

$$\begin{aligned} \langle x, y \rangle &= \quad x_0 y_0 + x_1 y_1 + \cdots + x_d y_d, \text{ and} \\ \langle x | y \rangle &= -x_0 y_0 + x_1 y_1 + \cdots + x_d y_d. \end{aligned}$$

Define M_κ^d, the d-dimensional space with constant curvature κ, as

$$M_\kappa^d = \left\{ x \in \mathbb{R}^{d+1} \mid \langle x, x \rangle = \frac{1}{\kappa} \right\}, \text{if } \kappa > 0,$$

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The geodesic distance is given by

$$d_\kappa(x, y) = \begin{cases} \frac{1}{\sqrt{+\kappa}} \operatorname{arccos}(\kappa \langle x, y \rangle), & \text{if } \kappa > 0, \\ \frac{1}{\sqrt{-\kappa}} \operatorname{arcosh}(\kappa \langle x | y \rangle), & \text{if } \kappa < 0. \end{cases}$$

Spherical and hyperbolic spaces

Notice that

$$\mathbf{diam}(M_\kappa^d) = \begin{cases} \frac{\pi}{\sqrt{\kappa}}, & \text{if } \kappa > 0, \\ \infty, & \text{if } \kappa \leq 0. \end{cases}$$

Spherical and hyperbolic spaces

Notice that

$$\mathbf{diam}(M_\kappa^d) = \begin{cases} \frac{\pi}{\sqrt{\kappa}}, & \text{if } \kappa > 0, \\ \infty, & \text{if } \kappa \leq 0. \end{cases}$$

Define the functions

$$c_\kappa(t) = \begin{cases} \cos\left(\sqrt{\kappa}t\right), & \text{if } \kappa > 0, \\ \cosh\left(\sqrt{-\kappa}t\right), & \text{if } \kappa < 0, \end{cases}$$

and

$$s_\kappa(t) = \begin{cases} \sin\left(\sqrt{\kappa}t\right), & \text{if } \kappa > 0, \\ \sinh\left(\sqrt{-\kappa}t\right), & \text{if } \kappa < 0. \end{cases}$$

Generalized Cayley-Menger determinant

Let $p_1, \ldots, p_n \in M_\kappa^d$. Define

$$\mathrm{CM}_\kappa(p_1, \ldots, p_n) = \begin{pmatrix} c_\kappa(d_{11}) & \cdots & c_\kappa(d_{1n}) \\ \vdots & \ddots & \vdots \\ c_\kappa(d_{n1}) & \cdots & c_\kappa(d_{nn}) \end{pmatrix},$$

and

$$P_\kappa(p_1, \ldots, p_n) = \begin{pmatrix} s_\kappa^2(d_{11}/2) & \cdots & s_\kappa^2(d_{1n}/2) \\ \vdots & \ddots & \vdots \\ s_\kappa^2(d_{n1}/2) & \cdots & s_\kappa^2(d_{nn}/2) \end{pmatrix}.$$

Denote their determinants by $\Delta_\kappa(p_1, \ldots, p_n) = \det(\mathrm{CM}_\kappa(p_1, \ldots, p_n))$ and $\gamma_\kappa(p_1, \ldots, p_n) = \det(P_\kappa(p_1, \ldots, p_n))$.

Properties of non-Euclidean Cayley-Menger determinants

- If $\kappa > 0$, $\mathrm{CM}_\kappa(p_1, \dots, p_n)$ is positive semi-definite.
- If $\kappa < 0$, $(-1)^{n+1}\Delta_\kappa(p_1, \dots, p_n) \geq 0$.
- If $n \geq d + 2$, $\Delta_\kappa(p_1, \dots, p_n) = 0$.

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- If $n \geq d + 2$, $\Delta_\kappa(p_1, \ldots, p_n) = 0$.

- Let $\kappa > 0$. Given a finite metric space (X, d_X), there exists an isometric embedding $X \hookrightarrow M_\kappa^n$ if and only if $\mathbf{diam}(X) \leq D_\kappa$ and there exist $p_1, \ldots, p_{n+1} \in X$ such that:
  - $(-1)^k\Delta_\kappa(p_1, \ldots, p_k) \geq 0$, for every $k = 2, \ldots, n+1$,
  - $\Delta_\kappa(p_1, \ldots, p_n, x) = \Delta_\kappa(p_1, \ldots, p_n, y) = \Delta_\kappa(p_1, \ldots, p_n, x, y) = 0$ for all $x, y \in X$.

  There is no direct analogue of this theorem in hyperbolic geometry. See Theorems 63.1 and 111.6 in [Blu70].

The Earth is round

Figure: Four cities on Earth: London, Los Angeles, Tokyo and Dubai.

See [Tao19] for further discussion. Map generated on `mapcustomizer.com`.

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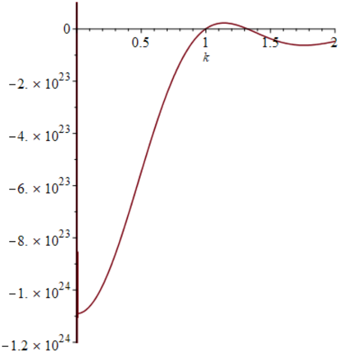

Figure: Sign of the determinant $\Delta_\kappa(P_1, P_2, P_3, P_4)$. [Tao19]

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Desnanot-Jacobi identity

Let M be an n-by-n matrix.

- Denote by $M_{\sim i,\sim j}$ the submatrix obtained by deleting the i-th row and j-th columns of M.
- If $i < j$ and $h < k$, let $M_{\sim i,j,\sim h,k} = (M_{\sim j,\sim k})_{\sim i,\sim h}$.

Theorem ([Gri20])

$$\det(M)\det(M_{\sim 1,n,\sim 1,n}) = \det(M_{\sim 1,\sim 1})\det(M_{\sim n,\sim n}) - \det(M_{\sim 1,\sim n})\det(M_{\sim n,\sim 1}).$$

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Theorem ([Gri20])

$$\det(M)\det(M_{\sim 1,n,\sim 1,n}) = \det(M_{\sim 1,\sim 1})\det(M_{\sim n,\sim n}) - \det(M_{\sim 1,\sim n})\det(M_{\sim n,\sim 1}).$$

This equation is also known as the Lewis Carroll identity.

Generalized Ptolemy's inequality

Let $p_1, \ldots, p_{n+2} \in M_\kappa^n$.

$$-\Delta_\kappa(p_1, \ldots, p_{n+2}) = \det \left(\begin{array}{c|ccc} -1 & 0 & \cdots & 0 \\ \hline 0 & c_\kappa(d_{11}) & \cdots & c_\kappa(d_{1,n+2}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & c_\kappa(d_{n+2,1}) & \cdots & c_\kappa(d_{n+2,n+2}) \end{array} \right)$$

Generalized Ptolemy's inequality

Let $p_1, \ldots, p_{n+2} \in M_\kappa^n$.

$$0 = \det \left(\begin{array}{c|ccc} -1 & 1 & \cdots & 0 \\ \hline 0 & c_\kappa(d_{11}) & \cdots & c_\kappa(d_{1,n+2}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & c_\kappa(d_{n+2,1}) & \cdots & c_\kappa(d_{n+2,n+2}) \end{array} \right)$$

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Let $p_1, \ldots, p_{n+2} \in M^n_\kappa$.

$$0 = \det \left(\begin{array}{c|ccc} -1 & 1 & \cdots & \color{red}{1} \\ \hline 0 & c_\kappa(d_{11}) & \cdots & c_\kappa(d_{1,n+2}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & c_\kappa(d_{n+2,1}) & \cdots & c_\kappa(d_{n+2,n+2}) \end{array} \right)$$

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Let $p_1, \ldots, p_{n+2} \in M^n_\kappa$.

$$0 = \det \left(\begin{array}{c|ccc} -1 & 1 & \cdots & 1 \\ \hline \color{red}{1} & c_\kappa(d_{11}) \color{red}{- 1} & \cdots & c_\kappa(d_{1,n+2}) \color{red}{- 1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & c_\kappa(d_{n+2,1}) & \cdots & c_\kappa(d_{n+2,n+2}) \end{array} \right)$$

Let $p_1, \ldots, p_{n+2} \in M^n_\kappa$.

$$0 = \det \left(\begin{array}{c|ccc} -1 & 1 & \cdots & 1 \\ \hline 1 & c_\kappa(d_{11}) - 1 & \cdots & c_\kappa(d_{1,n+2}) - 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & c_\kappa(d_{n+2,1}) - 1 & \cdots & c_\kappa(d_{n+2,n+2}) - 1 \end{array} \right)$$

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By the sum of angles formula, $c_\kappa(t) - 1 = -2\varepsilon_\kappa s_\kappa^2(t/2)$, where $\varepsilon_\kappa = \operatorname{sgn}(\kappa)$.

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$$\begin{aligned} &\det(M)\det(M_{\sim 1,n,\sim 1,n}) \\ &= \det(M_{\sim 1,\sim 1})\det(M_{\sim n,\sim n}) - \det(M_{\sim 1,\sim n})\det(M_{\sim n,\sim 1}). \end{aligned}$$

Generalized Ptolemy's inequality

Let $p_1, \ldots, p_{n+2} \in M_\kappa^n$.

$$0 = \det \left(\begin{array}{c|ccc} -1 & 1 & \cdots & 1 \\ \hline 1 & -2\varepsilon_\kappa s_\kappa^2(d_{11}/2) & \cdots & -2\varepsilon_\kappa s_\kappa^2(d_{1,n+2}/2) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & -2\varepsilon_\kappa s_\kappa^2(d_{n+2,1}/2) & \cdots & -2\varepsilon_\kappa s_\kappa^2(d_{n+2,n+2}/2) \end{array} \right)$$

$$\begin{aligned} 0 &= \det(M) \det(M_{\sim 1,n,\sim 1,n}) \\ &= \det(M_{\sim 1,\sim 1}) \det(M_{\sim n,\sim n}) - \det(M_{\sim 1,\sim n}) \det(M_{\sim n,\sim 1}). \end{aligned}$$

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$$\begin{aligned} 0 &= \det(M)\det(M_{\sim 1,n,\sim 1,n}) \\ &= \det(M_{\sim 1,\sim 1})\det(M_{\sim n,\sim n}) - \det(M_{\sim 1,\sim n})^2. \end{aligned}$$

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$$\begin{aligned} 0 = &\ \det(M)\det(M_{\sim 1,n,\sim 1,n}) \\ = &\ (-2\varepsilon_\kappa)^{n+2}\gamma_\kappa(p_1, \ldots, p_{n+2})\det(M_{\sim n,\sim n}) - \det(M_{\sim 1,\sim n})^2. \end{aligned}$$

Generalized Ptolemy's inequality

Let $p_1, \ldots, p_{n+2} \in M^n_\kappa$.

$$0 = \det \left(\begin{array}{c|ccc} -1 & 1 & \cdots & 1 \\ \hline 1 & -2\varepsilon_\kappa s_\kappa^2(d_{11}/2) & \cdots & -2\varepsilon_\kappa s_\kappa^2(d_{1,n+2}/2) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & -2\varepsilon_\kappa s_\kappa^2(d_{n+2,1}/2) & \cdots & -2\varepsilon_\kappa s_\kappa^2(d_{n+2,n+2}/2) \end{array} \right)$$

$$\begin{aligned} 0 &= \det(M)\det(M_{\sim 1,n,\sim 1,n}) \\ &= -(-2\varepsilon_\kappa)^{n+2}\gamma_\kappa(p_1,\ldots,p_{n+2})\Delta_\kappa(p_1,\ldots,p_{n+1}) - \det(M_{\sim 1,\sim n})^2. \end{aligned}$$

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$$\begin{aligned} 0 = &\ \det(M)\det(M_{\sim 1,n,\sim 1,n}) \\ = &-(-2\varepsilon_\kappa)^{n+2}\gamma_\kappa(p_1, \ldots, p_{n+2})\Delta_\kappa(p_1, \ldots, p_{n+1}) - \det(M_{\sim 1,\sim n})^2. \end{aligned}$$

Theorem

$\gamma_\kappa(p_1, \ldots, p_{n+2})$ is 0 or has sign $(-1)^{n+1}$.

This theorem was proved for $n = 4$ in [Val70a] (for $\kappa > 0$) and [Val70b] (for $\kappa < 0$).

Ptolemy's inequality in non-Euclidean geometry

If $n = 2$, $P(p_1, p_2, p_3, p_4) = \det \begin{pmatrix} d_{11}^2 & \cdots & d_{14}^2 \\ \vdots & \ddots & \vdots \\ d_{41}^2 & \cdots & d_{44}^2 \end{pmatrix}$ factors as

$$\begin{aligned} P(p_1, p_2, p_3, p_4) = -&(d_{13}d_{24} + d_{12}d_{34} + d_{23}d_{41}) \\ &\cdot (-d_{13}d_{24} + d_{12}d_{34} + d_{23}d_{41}) \\ &\cdot (+d_{13}d_{24} - d_{12}d_{34} + d_{23}d_{41}) \\ &\cdot (+d_{13}d_{24} + d_{12}d_{34} - d_{23}d_{41}). \end{aligned}$$

Ptolemy's inequality in non-Euclidean geometry

Let $s_{ij} = s_\kappa(d_{ij}/2)$. If $n = 2$, $\gamma_\kappa(p_1, p_2, p_3, p_4) = \det \begin{pmatrix} s_{11}^2 & \cdots & s_{14}^2 \\ \vdots & \ddots & \vdots \\ s_{41}^2 & \cdots & s_{44}^2 \end{pmatrix}$ factors as

$$\begin{aligned}\gamma_\kappa(p_1, p_2, p_3, p_4) = -&[s_{13}s_{24} + s_{12}s_{34} + s_{23}s_{41}] \\ &\cdot [-s_{13}s_{24} + s_{12}s_{34} + s_{23}s_{41}] \\ &\cdot [+s_{13}s_{24} - s_{12}s_{34} + s_{23}s_{41}] \\ &\cdot [+s_{13}s_{24} + s_{12}s_{34} - s_{23}s_{41}]\end{aligned}$$

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Ptolemy's inequality in non-Euclidean geometry

Let $s_{ij} = s_\kappa(d_{ij}/2)$. If $n = 2$, $\gamma_\kappa(p_1, p_2, p_3, p_4) = \det \begin{pmatrix} s_{11}^2 & \cdots & s_{14}^2 \\ \vdots & \ddots & \vdots \\ s_{41}^2 & \cdots & s_{44}^2 \end{pmatrix}$ factors as

$$\begin{aligned} \gamma_\kappa(p_1, p_2, p_3, p_4) = -[&s_{13}s_{24} + s_{12}s_{34} + s_{23}s_{41}] \\ &\cdot [-s_{13}s_{24} + s_{12}s_{34} + s_{23}s_{41}] \\ &\cdot [+s_{13}s_{24} - s_{12}s_{34} + s_{23}s_{41}] \\ &\cdot [+s_{13}s_{24} + s_{12}s_{34} - s_{23}s_{41}] \leq 0. \end{aligned}$$

Theorem (Ptolemy's Theorem [Val70a, Val70b])

Let $\kappa \neq 0$. For any $p_1, p_2, p_3, p_4 \in M_\kappa^2$,

$$s_\kappa(d_{13}/2)s_\kappa(d_{24}/2) \leq s_\kappa(d_{12}/2)s_\kappa(d_{34}/2) + s_\kappa(d_{23}/2)s_\kappa(d_{41}/2).$$

Overview

Graphs and trees

Definition

A tree is a graph with no cycles.

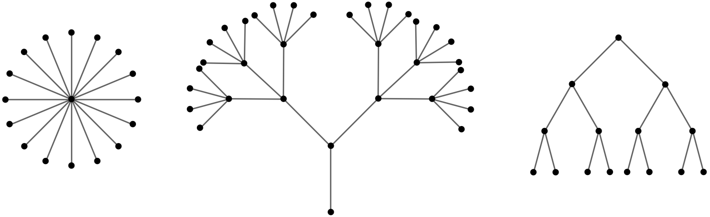

Figure: A collection of trees. Image from https://ptwiddle.github.io/MAS341-Graph-Theory-2017/lecturenotes/lecture6.html.

Metric trees

4-point condition

Let X be a finite metric space. If

$$d_{13} + d_{24} \leq \max\{d_{12} + d_{34}, d_{23} + d_{41}\}$$

for every $x_1, x_2, x_3, x_4 \in X$, there exists a metric tree T such that $X \hookrightarrow T$.

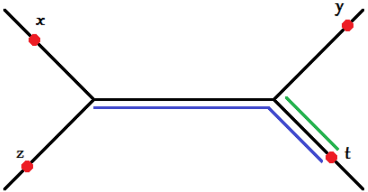

Figure: Image from:
https://commons.wikimedia.org/wiki/File:Four_point_condition.png

Embeddings into trees

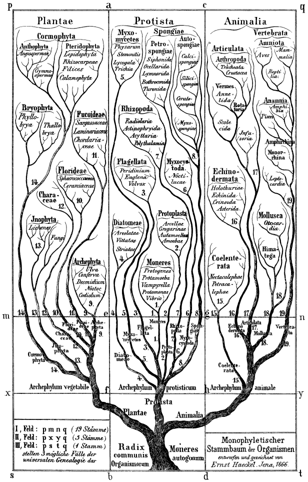

Figure: A tree of life by Haeckel. Image from
`https://en.wikipedia.org/wiki/File:Haeckel_arbol_bn.png`

Ptolemy's inequality and the 4-point condition

4-point condition:

$$d_{13} + d_{24} \leq \max\{d_{12} + d_{34}, d_{23} + d_{41}\}$$

Ptolemy's inequality:

$$d_{13} \cdot d_{24} \leq d_{12} \cdot d_{34} + d_{23} \cdot d_{41}.$$

Ptolemy's inequality and the 4-point condition

4-point condition:

$$d_{13} + d_{24} \leq \max\{d_{12} + d_{34}, d_{23} + d_{41}\}$$

Ptolemy's inequality:

$$d_{13} \cdot d_{24} \leq d_{12} \cdot d_{34} + d_{23} \cdot d_{41}.$$

Question

Is there a link between these inequalities?

CAT spaces and hyperbolicity

- A CAT(κ) space is a geodesic metric space where geodesic triangles are "thinner" than those in M_κ^2.
- A metric space (X, d_X) is called δ-*hyperbolic* if

$$d_{13} + d_{24} \leq \max(d_{12} + d_{34}, d_{23} + d_{41}) + 2\delta \qquad (1)$$

  for all $x_1, x_2, x_3, x_4 \in X$. Define the *hyperbolicity* of X, denoted by $\mathrm{hyp}(X)$, as the infimal δ such that (1) holds.
- An $\mathbb{R}$-*tree* is a geodesic metric space with hyperbolicity 0.
- Metric trees are locally finite $\mathbb{R}$-trees.

CAT spaces and hyperbolicity

Facts:

- Let $\kappa < 0$. Every CAT(κ) space X is δ-hyperbolic for $\delta = \ln(2)/\sqrt{-\kappa}$.
- For all $\kappa \in \mathbb{R}$, the κ-Ptolemy inequality holds in CAT(κ).
- The class of $\mathbb{R}$-trees coincides with the geodesic metric spaces that are CAT(κ) for all $\kappa \in \mathbb{R}$.

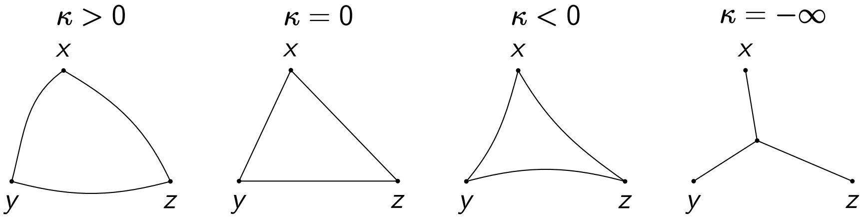

Figure: Triangles in spaces with constant curvature.

Ptolemy's inequality for metric trees

If $a, b, c, d \geq 0$ and $c + d \leq a + b$,

$$\lim_{\kappa \to -\infty} \frac{2}{\sqrt{-\kappa}} \operatorname{arcsinh}\left[\sinh\left(\frac{\sqrt{-\kappa}}{2}a\right) \cdot \sinh\left(\frac{\sqrt{-\kappa}}{2}b\right) + \sinh\left(\frac{\sqrt{-\kappa}}{2}c\right) \cdot \sinh\left(\frac{\sqrt{-\kappa}}{2}d\right)\right]$$

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$$= \lim_{\kappa \to -\infty} \frac{2}{\sqrt{-\kappa}} \ln\left[\sinh\left(\frac{\sqrt{-\kappa}}{2}a\right) \cdot \sinh\left(\frac{\sqrt{-\kappa}}{2}b\right) + \sinh\left(\frac{\sqrt{-\kappa}}{2}c\right) \cdot \sinh\left(\frac{\sqrt{-\kappa}}{2}d\right)\right]$$

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$$= \lim_{\kappa \to -\infty} \frac{2}{\sqrt{-\kappa}} \ln\left[\frac{\exp\left(\frac{\sqrt{-\kappa}}{2}a\right) - \exp\left(-\frac{\sqrt{-\kappa}}{2}a\right)}{2} \cdot \frac{\exp\left(\frac{\sqrt{-\kappa}}{2}b\right) - \exp\left(-\frac{\sqrt{-\kappa}}{2}b\right)}{2} + \frac{\exp\left(\frac{\sqrt{-\kappa}}{2}c\right) - \exp\left(-\frac{\sqrt{-\kappa}}{2}c\right)}{2} \cdot \frac{\exp\left(\frac{\sqrt{-\kappa}}{2}d\right) - \exp\left(-\frac{\sqrt{-\kappa}}{2}d\right)}{2}\right]$$

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If $a, b, c, d \geq 0$ and $c + d \leq a + b$,

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$$= \lim_{\kappa \to -\infty} \frac{2}{\sqrt{-\kappa}} \ln\left[\exp\left(\frac{\sqrt{-\kappa}}{2}(a+b)\right)\left(\frac{1-\exp(-\sqrt{-\kappa}a)}{2} \cdot \frac{1-\exp(-\sqrt{-\kappa}b)}{2} + \frac{\exp\left(\frac{\sqrt{-\kappa}}{2}(c+d)\right)}{\exp\left(\frac{\sqrt{-\kappa}}{2}(a+b)\right)} \cdot \frac{1-\exp(-\sqrt{-\kappa}c)}{2} \cdot \frac{1-\exp(-\sqrt{-\kappa}d)}{2}\right)\right]$$

Ptolemy's inequality for metric trees

If $a, b, c, d \geq 0$ and $c + d \leq a + b$,

$$\lim_{\kappa \to -\infty} \frac{2}{\sqrt{-\kappa}} \operatorname{arcsinh}\left[\sinh\left(\frac{\sqrt{-\kappa}}{2}a\right) \cdot \sinh\left(\frac{\sqrt{-\kappa}}{2}b\right) + \sinh\left(\frac{\sqrt{-\kappa}}{2}c\right) \cdot \sinh\left(\frac{\sqrt{-\kappa}}{2}d\right)\right]$$

$$= \lim_{\kappa \to -\infty} \frac{2}{\sqrt{-\kappa}} \ln\left[\exp\left(\frac{\sqrt{-\kappa}}{2}(a+b)\right)\right]$$

$$+ \lim_{\kappa \to -\infty} \frac{2}{\sqrt{-\kappa}} \ln\left[\frac{1-\exp(-\sqrt{-\kappa}a)}{2} \cdot \frac{1-\exp(-\sqrt{-\kappa}b)}{2} + \frac{\exp\left(\frac{\sqrt{-\kappa}}{2}(c+d)\right)}{\exp\left(\frac{\sqrt{-\kappa}}{2}(a+b)\right)} \cdot \frac{1-\exp(-\sqrt{-\kappa}c)}{2} \cdot \frac{1-\exp(-\sqrt{-\kappa}d)}{2}\right]$$

Ptolemy's inequality for metric trees

If $a, b, c, d \geq 0$ and $c + d \leq a + b$,

$$\lim_{\kappa \to -\infty} \frac{2}{\sqrt{-\kappa}} \operatorname{arcsinh}\left[\sinh\left(\frac{\sqrt{-\kappa}}{2}a\right) \cdot \sinh\left(\frac{\sqrt{-\kappa}}{2}b\right) + \sinh\left(\frac{\sqrt{-\kappa}}{2}c\right) \cdot \sinh\left(\frac{\sqrt{-\kappa}}{2}d\right)\right]$$
$$= a + b$$

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$$= a + b = \max(a + b, c + d).$$

Ptolemy's inequality for metric trees

If $a, b, c, d \geq 0$ and $c + d \leq a + b$,

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$$= a + b = \max(a + b, c + d).$$

Thus, if $p_1, p_2, p_3, p_4 \in X$ satisfy

$$s_\kappa(d_{13}/2)s_\kappa(d_{24}/2) \leq s_\kappa(d_{12}/2)s_\kappa(d_{34}/2) + s_\kappa(d_{23}/2)s_\kappa(d_{41}/2)$$

for all κ, the above calculation gives:

Ptolemy's inequality for metric trees

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for all κ, the above calculation gives:

4-point condition

$$d_{13} + d_{24} \leq \max(d_{12} + d_{34}, d_{23} + d_{41}).$$

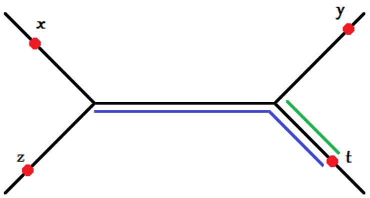
This calculation was obtained joint with F. Mémoli. See [GM21].

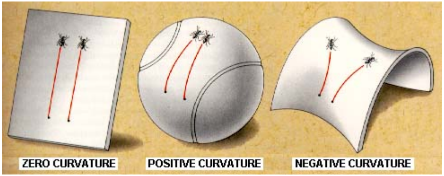
Summary

- Ptolemy's inequality is a relationship that involves the distances between 4 points in the Euclidean plane.
- It is attributed to Claudius Ptolemy (c. 100-170 AD), who used it, among other mathematical innovations, to construct the most thorough and precise table of chords of his time.
- Ptolemy's Almagest was an authoritative text for over a millennium.

Summary

- Ptolemy's inequality can be generalized to non-Euclidean geometries. The proof involves properties of the Cayley-Menger determinant and a matrix identity known as the Lewis Carroll identity.
- The degenerate case of Ptolemy's inequality in $\kappa = -\infty$ is the 4-point condition, an inequality that characterizes $\mathbb{R}$-trees.





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Questions

Thanks for your attention!

Properties of the Spherical Cayley-Menger determinant

Let $\kappa > 0$. Recall that $d_{ij} = \frac{1}{\sqrt{\kappa}} \arccos(\sqrt{\kappa}\langle p_i, p_j \rangle)$, so

$$c_\kappa(d_{ij}) = \cos(\sqrt{\kappa} d_{ij}) = \sqrt{\kappa}\langle p_i, p_j \rangle.$$

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Then:

$$\mathrm{CM}_\kappa(p_1, \dots, p_n) = \sqrt{\kappa} \begin{pmatrix} \langle p_1, p_1\rangle & \cdots & \langle p_1, p_n\rangle \\ \vdots & \ddots & \vdots \\ \langle p_n, p_1\rangle & \cdots & \langle p_n, p_n\rangle \end{pmatrix}$$

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where $\Sigma = \left(p_1 | \cdots | p_n\right) \in \mathbb{R}^{d+1 \times n}$. Thus:

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- $\Delta_\kappa(p_1, \dots, p_n) \geq 0$.

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Then:

$$\mathrm{CM}_\kappa(p_1, \dots, p_n) = \sqrt{\kappa} \begin{pmatrix} \langle p_1, p_1 \rangle & \cdots & \langle p_1, p_n \rangle \\ \vdots & \ddots & \vdots \\ \langle p_n, p_1 \rangle & \cdots & \langle p_n, p_n \rangle \end{pmatrix} = \sqrt{\kappa}\Sigma^T \Sigma,$$

where $\Sigma = \left(p_1 | \cdots | p_n\right) \in \mathbb{R}^{d+1 \times n}$. Thus:

- $\Delta_\kappa(p_1, \dots, p_n) \geq 0$.
- If $n \geq d+2$, $\mathrm{rk}(\mathrm{CM}_\kappa(p_1, \dots, p_n)) \leq d-1 < n$, so $\Delta_\kappa(p_1, \dots, p_n) = 0$.

Properties of the Hyperbolic Cayley-Menger determinant

With some more work, we can show analogous properties for $\kappa < 0$, so that:

- $\Delta_\kappa(p_1, \ldots, p_n)$ is 0 or has sign $(\mathrm{sgn}(\kappa))^{n+1}$.
- If $n \geq d + 2$, $\Delta_\kappa(p_1, \ldots, p_n) = 0$.

CAT(κ) spaces

Proposition ([BH09])

Let X be a complete metric space. Then X is a CAT(κ) *space if, and only if, two conditions are satisfied:*

- *Every pair $x, x' \in X$ with $d_X(x, x') < D_\kappa$ has approximate midpoints.*
- *For every 4-tuple $(x_1, y_1, x_2, y_2) \in X$ such that $d_X(x_1, y_1) + d_X(y_1, x_2) + d_X(x_2, y_2) + d(y_2, x_1) < 2D_\kappa$, there exists a 4-tuple $(\overline{x}_1, \overline{y}_1, \overline{x}_2, \overline{y}_2) \in M_\kappa^2$ such that:*

$$d_X(x_i, y_j) = d_{M_\kappa^2}(\overline{x}_i, \overline{y}_j) \text{ for } i, j \in \{1, 2\},$$
$$d_X(x_1, x_2) \le d_{M_\kappa^2}(\overline{x}_1, \overline{x}_2), \text{ and}$$
$$d_X(y_1, y_2) \le d_{M_\kappa^2}(\overline{y}_1, \overline{y}_2).$$

Ptolemy's inequality in CAT(κ) spaces

Theorem

Let X be a CAT(κ) *space and $p_1, p_2, p_3, p_4 \in X$ such that $d_{12} + d_{23} + d_{34} + d_{41} < 2D_\kappa$. Then*

$$s_\kappa(d_{13}/2)s_\kappa(d_{24}/2) \leq s_\kappa(d_{12}/2)s_\kappa(d_{34}/2) + s_\kappa(d_{41}/2)s_\kappa(d_{23}/2).$$

By the CAT(κ) 4-point condition, there exist $\overline{p}_i \in M^2_\kappa$ with

$$\begin{aligned} d_{i,i+1} &= \overline{d}_{i,i+1}, \text{ and} \\ d_{i,i+2} &\leq \overline{d}_{i,i+2}. \end{aligned}$$

Then:

$$\begin{aligned} s_\kappa(d_{13}/2)s_\kappa(d_{24}/2) &\leq s_\kappa(\overline{d}_{13}/2)s_\kappa(\overline{d}_{24}/2) \\ &\leq s_\kappa(\overline{d}_{12}/2)s_\kappa(\overline{d}_{34}/2) + s_\kappa(\overline{d}_{23}/2)s_\kappa(\overline{d}_{41}/2) \\ &= s_\kappa(d_{12}/2)s_\kappa(d_{34}/2) + s_\kappa(d_{23}/2)s_\kappa(d_{41}/2). \end{aligned}$$