What is Amenability? 7/12/2022

Let G be a group, f a function from G to $\mathbb{R}$.
- To avoid issues of convergence, assume f is bounded

What does it mean to take the average value of f over G?

If G is finite then $\frac{1}{|G|} \sum_{g\in G} f(g)$

If $G = \mathbb{Z}$ then $\limsup_{n\to\infty} \frac{1}{2n} \sum_{i=-n}^{n} f(i)$

Seems reasonable. What about $G = S^1$?

In general, if m(f) denotes the average value (mean) of f then we want:

$m(f+h) = m(f) + m(h)$
$m(rf) = r\cdot m(f) \ \forall r\in\mathbb{R}$
$m(1_G) = 1$
If $f(g) \geq 0 \ \forall g\in G$ then $m(f) \geq 0$
If $\tilde{f}(g) = f(h^{-1}g)$ for some $h\in G$ then $m(\tilde{f}) = m(f)$

Every abelian group is amenable, but with this definition, proof requires analysis (weak* stuff)

Fact

Subgroups of Amenable groups are Amenable since a function on H gives a function on G
Pick coset representatives $S \subseteq G$ such that $G = \bigsqcup_{x \in S} xH$, then define $\tilde{f}(g) = f(x^{-1}g)$
such that $g \in xH$

Fact

If $G_1 < G_2 < \cdots$ $\bigcup G_i = G$ and G_i are all amenable, then G is amenable
(In general, direct limits of amenable are amenable)
If $(m_i)_{i \in \mathbb{N}}$ are means for $(G_i)_{i \in \mathbb{N}}$ then

$\tilde{m}_i(f) = m_i(f|_{G_i})$ gives a sequence of functions which has a convergent subsequence (because the closed unit ball in the dual of ℓ^∞ is compact in the weak* topology)

Fact: Quotients of Amenable are amenable

The existance of a mean is the same as the existance of a measure μ on G by putting $\mu(A) = m(\mathbb{1}_A)$ and $m(f) = \int f \, d\mu$

A Følner sequence is a sequence of finite subsets $F_1 \subseteq F_2 \subseteq \cdots$ s.t.,

$$\frac{|gF_n \Delta F_n|}{|F_n|} \xrightarrow{n \to \infty} 0$$

(replace $|\cdot|$ by μ for compact groups)

Given a Følner sequence define

$$\mu(A) = \lim_{n\to\infty} \frac{|A \cap F_n|}{|F_n|}$$

ultrafilter business to make sure limit exists

In fact amenability $\Longleftrightarrow$ Følner
but we don't need the forward direction

$\mathbb{Z}$ has Følner sequence $F_n = \{-n, \dots, n\}$

Any finitely generated abelian group is amenable
~~generated by $x_1, \dots, x_n$~~
since it is the quotient of $\mathbb{Z}^n$
which is amenable

(Any abelian is amenable by taking direct limits)

Solvable means amenable

What isn't amenable?

F_2 isn't!

Suppose F_2 has a measure μ

$$F_2 = A^+ \sqcup A^- \sqcup (B^+ \setminus \langle b \rangle) \sqcup (B^- \cup \langle b \rangle)$$

$$= A^+ \sqcup aA^-$$

$$= b^{-1}(B^+ \setminus \langle b \rangle) \sqcup (B^- \cup \langle b \rangle)$$

So $\mu(F_2) = \mu(A^+ \sqcup A^-) + \mu(B^+_{\langle b \rangle} \sqcup B^-_{\langle b \rangle})$

$$= \mu(A^+) + \mu(aA^{-1}) + \mu(B^+) + \mu(B^{-1})$$

$$= \mu(F_2) \qquad + \mu(F_2)$$

So any group containing F_2 is not amenable

A group is called paradoxical if there are $A_1, \ldots, A_n, B_1, \ldots, B_m$ pairwise disjoint such that

$$G = (\sqcup A_i) \sqcup (\sqcup B_j)$$

$$= \sqcup a_i A_i$$

$$= \sqcup b_j B_j$$

or equivalently $G = \sqcup A_i$

$$= \sqcup B_i$$

$$= (\sqcup a_i A_i) \sqcup (\sqcup b_i B_i)$$

If G is not amenable, then G is Paradoxical

G doesn't have a Følner sequence

- No Følner
- $|KF| \geq 2|F|$
- 2 to 1 $\varphi : G \to G$
- $\varphi(g) \in kg$
- Paradoxical

G has a Følner sequence iff $\forall$ finite $K \subseteq G$ $\forall \varepsilon > 0$, there is a finite F s.t.

$$\frac{|F \setminus kF|}{|F|} < \varepsilon \quad \forall k \in K$$

So there is $K_0 \subseteq G$, $\varepsilon_0 > 0$ s.t. for any finite $F \subseteq G$ there is $k \in K$ s.t.

$$\frac{|F \setminus kF|}{|F|} \geq \varepsilon_0$$

Let $K_1 = K_0 \cup \{e\}$

Then $|K_1 F| - |F| = |K_1 F \setminus F| = |K_0 F \setminus F|$

$\geq |F \setminus k F| \geq \varepsilon_0 |F|$

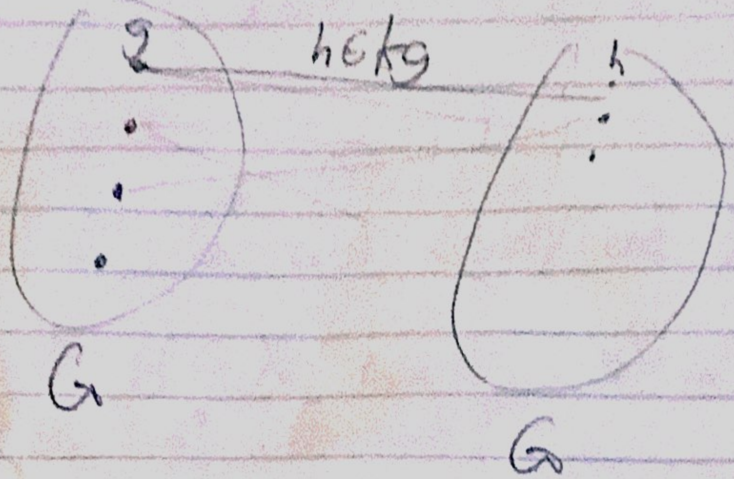
$k \in K_0$

so $|K_1 F| \geq (1 + \varepsilon_0)|F|$

Let $K = K_1^n$ for large enough n so that $|KF| \geq 2|F|$

$k \in K$ $\quad \bigsqcup_{k \in K}$

Consider the set of all pairs (g,h) such that $h \in Kg$



$|KF| \geq 2|F|$

Hall's marriage theorem gives
2-to-1 function $\varphi : G \to G$ such that
surjective
$\varphi(g) \in Kg$

(AoC)

Let ψ_1 and ψ_2 be such that

$\psi_1(g) \neq \psi_2(g)$ $\forall g$, $\varphi \circ \psi_1 = \varphi \circ \psi_2 = id$

Then $\theta(g) = \psi_1(g) g^{-1} \in K$

$\theta_2(g) = \psi_2(g) g^{-1} \in K$

$\theta_i : G \twoheadrightarrow K$

$A_k = \theta_1^{-1}(k)$, $B_k = \theta_2^{-1}(k)$

$G = \bigsqcup_{k \in K} A_k = \bigsqcup_{k \in K} B_k$

$$G = \psi_1(G) \sqcup \psi_2(G)$$
$$= \left(\bigsqcup_k A_k\right) \sqcup \left(\bigsqcup_k B_k\right)$$

Let $\overset{\text{Paradoxical}}{G}$ act freely on $X \neq \emptyset$

Let $R \subseteq X$ contain exactly one Point from each G-orbit

Then $X = \left(\bigsqcup A_i R\right) \sqcup \left(\bigsqcup B_j \cdot R\right)$
$$= \bigsqcup a_i A_i \cdot R$$
$$= \bigsqcup b_j B_j \cdot R$$

$SO(3)$ is not amenable

$$\left\langle \begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix} \right\rangle$$

is free

For $g \in SO_3(\mathbb{Q})$ Let X_g and Y_g $\in S^2$ denote the points fixed by g

Then $SO_3(\mathbb{Q})$ acts freely on

$S^2 \setminus \{X_g, Y_g \mid g \in SO_3(\mathbb{Q})\}$

So this space is Paradoxical.

Sources

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