Notations

- Given a matrix A we will say
  - $A > 0$ if each entry of A is positive.
  - $A \geqslant 0$ —— " —— " —— " non-negative.
- $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- $\pi_K : \mathbb{R}^K \twoheadrightarrow \mathbb{R}$
  $(x_i)_{i=1}^K \longmapsto x_K$
- e_i is i^{th} unit vector in $\mathbb{R}^K$ and $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.
- Given $K \geqslant 2$, $\vec{x}, \vec{y} \in \mathbb{R}^K$ we write $\vec{x} \overset{p}{=} \vec{y}$ iff $\exists\, c \neq 0$ s.t. $\vec{x} = c\vec{y}$.

1D continued fractions (C.f.)

Expression $x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n + \cfrac{1}{\ddots}}}}$

(also written as $x = [a_0; a_1, a_2, \ldots, a_n, \ldots]$)

are 1D C.f. Here, $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}_0$.

For $m \geq 0$, $\frac{p_m}{q_m} = [a_0; a_1, \ldots, a_m]$ is called

m-th convergent; which has properties.

Properties:

(I) $\lim_{m \to \infty} \frac{p_m}{q_m} = x$.

(II) $p_0 = a_0$, $p_1 = a_0 a_1 + 1$, $p_n = p_{n-1} a_n + p_{n-2}$

for $n \geq 2$

(III) $q_0 = 1,\ q_1 = a_1,\ \ q_n = q_{n-1}a_n + q_{n-2}$ for $n \geq 2$.

(IV) $\forall n \in \mathbb{N},\ \ p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$.

Simple (1D) continued Fraction, in terms of Matrices

For $m \in \mathbb{N}_0$, $D_1[m] := \begin{bmatrix} 0 & 1 \\ 1 & m \end{bmatrix} \in GL(2,\mathbb{Z})$

and note that,

$$\begin{pmatrix} q_{m-1} & q_m \\ p_{m-1} & p_m \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_m \end{pmatrix}$$

so,

$$\begin{pmatrix} q_m \\ p_m \end{pmatrix} = \begin{pmatrix} q_{m-1} & q_m \\ p_{m-1} & p_m \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & a_0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \times \cdots \times \begin{pmatrix} 0 & 1 \\ 1 & a_m \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Also,
$$\begin{pmatrix} q_m \\ p_m \end{pmatrix} \overset{p}{=} \begin{pmatrix} 1 \\ \frac{p_m}{q_m} \end{pmatrix}$$
So,

$$\begin{pmatrix} 1 \\ x \end{pmatrix} = \lim_{m \to \infty} \begin{pmatrix} 1 \\ \frac{p_m}{q_m} \end{pmatrix}$$

$$\overset{p}{=} \lim_{m \to \infty} \left[D_1(a_0) \cdots D_1[a_m] \times \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

ie. x has 1D c.f. $x = [a_0; a_1, \dots]$ iff

$$\lim_{m\to\infty}\left(D_1[a_0]\times D_1[a_1]\times\cdots\times D_1[a_m]\right)\begin{pmatrix}0\\1\end{pmatrix} \overset{p}{=} \begin{pmatrix}1\\x\end{pmatrix}$$

How to transition from matrix to 1 dimensional continued fraction?

For any $m\in\mathbb{N}_0$, define $C_1[m]:\overline{\mathbb{R}}\longrightarrow\overline{\mathbb{R}}$

such that $C_1[m](x) = m+\frac{1}{x}$, $\forall x\in\overline{\mathbb{R}}$.

A simple computation yields that

$$D_1[m]\begin{pmatrix}1\\x\end{pmatrix} \overset{p}{=} \begin{pmatrix}1\\C_1[m](x)\end{pmatrix}$$

<u>Easy Lemma</u>: $x\in\mathbb{R}$ has 1dimensional c.f.

$x=[a_0;a_1,a_2,\ldots]$ if and only if

$$\boxed{x \overset{p}{=} \lim_{m\to\infty} C_1[a_0]\times C_1[a_1]\times\cdots\times C_1[a_m](\infty)}$$

Euler, Lagrange & Periodic 1D continued fractions

> Euler proved that if $x \in \mathbb{R}$ has an eventually periodic 1D continued fraction, then x must be a quadratic irrational.

- Eventually periodic means c.f. of the form

$$[a_0, a_1, \ldots, a_s, \underbrace{\overline{a_{s+1}, \ldots, a_{s+p}}}_{\text{repeating}}]$$

for some $s \geqslant 0$ and $p > 0$.

> Lagrange proved the converse of Euler's result, which established that

$x \in \mathbb{R}$ has eventually periodic continued fraction iff x is a quadratic irrational.

Multidimensional continued fractions are alogrithms that attempt to generalize this fact to $\mathbb{R}^n$, analogously.

One example of such desired generalization is the following statement:

Let $n \geq 2$ & $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$. We want to associate to $\vec{x}$, a sequence of vectors in $\mathbb{R}^n$, say $\left\{ \vec{y}^{(i)} \right\}_{i=1}^{\infty}$ such that the sequence is eventually periodic iff both the following are satisfied:

(I) $x_1, x_2, \ldots, x_n$ are irrationals of degree at most $(n+1)$.

(II) The set $\{1, x_1, x_2, \ldots, x_n\}$ is $\mathbb{Q}$-linearly independent.

we will present one such n-dimensional continued fraction algorithm, which occurs as a natural generalization of matrix form of 1D continued fraction.

Defining Multidimensional Continued Fractions

> Given $n \geq 1$ and $\vec{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{N}_0^n$

define

$$D_n[\vec{y}] := \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & & 0 & y_1 \\ 0 & 1 & & 0 & y_2 \\ \vdots & 0 & & \vdots & \vdots \\ \vdots & \vdots & & 0 & y_{n-1} \\ 0 & 0 & & 1 & y_n \end{pmatrix} \in GL(n+1, \mathbb{R})$$

we say that $w \in \mathbb{R}^n$ has n-dimensional continued fraction $[\vec{y}^{(1)}, \vec{y}^{(2)}, \ldots]$ for some sequence $\{\vec{y}^{(i)}\}_{i=1}^{\infty} \subseteq \mathbb{N}_0^n$ iff

$$\begin{pmatrix} 1 \\ \vartheta \end{pmatrix} \overset{p}{=} \lim_{m \to \infty} \left(\left(\prod_{i=1}^{m} D_n\left[\vec{y}^{(i)}\right] \right) \begin{pmatrix} 0 \\ e_n \end{pmatrix} \right)$$

<u>Caution</u>:

(1) Not every $\vartheta \in \mathbb{R}^n$ has a n-dimensional continued fraction.

(2) Not every $\left(\vec{y}^{(i)}\right)_{i=1}^{\infty} \in \mathbb{N}_0^n$ yields a limit in

$$\lim_{m \to \infty} \left(\left(\prod_{i=1}^{m} D_n\left[\vec{y}^{(i)}\right] \right) \begin{pmatrix} 0 \\ e_n \end{pmatrix} \right)$$

We just generalized D_1 to D_n. Similarly, we can generalize the map G_1 to G_n.

For any $\vec{y} = (y_1, y_2, \dots, y_n) \in \mathbb{N}_0^n$, define

$$C_n[\vec{y}] : A \subseteq (\overline{\mathbb{R}})^n \longrightarrow \mathbb{R}^n$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} y_1 + \frac{1}{x_n} \\ y_2 + x_1/x_n \\ \vdots \\ y_n + \frac{x_{n-1}}{x_n} \end{pmatrix}$$

<u>A simple computation can verify</u>:

$$D_n[\vec{y}] \begin{pmatrix} 1 \\ \omega \end{pmatrix} \overset{p}{=} \begin{pmatrix} 1 \\ C_n[\vec{y}](\omega) \end{pmatrix}$$

Now, we want to prove a theorem; we need some definitions:

Perron-Frobenius Matrices

A matrix $M \in GL(n, \mathbb{R})$ is called P-F if $M \geq 0$ and M is not of the form

$$\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}.$$

Theorem (Bauer, M. (1992)):

Let $n \geq 1$, M be a P-F matrix such that for some $p \in \mathbb{N}$ and $\vec{y}^{(1)}, \vec{y}^{(2)}, \ldots, \vec{y}^{(p)} \in \mathbb{N}_0^{\,n}$

$$M = D_n[\vec{y}^{(1)}] \cdot D_n[\vec{y}^{(2)}] \cdot \cdots \cdot D_n[\vec{y}^{(p)}].$$

Then,
the spectral radius λ and eigenvector $x^* = (1 \quad \vec{\vartheta})^T$ associated with λ are given as follows:

① $\vec{\vartheta} = \overline{[\vec{y}^{(1)}, \vec{y}^{(2)}, \ldots, \vec{y}^{(p)}]}$ and

(11) $\lambda = \prod_{\nu=1}^{p} \pi_n \left(\sigma^{\nu-1} \overline{[\vec{y}^{(1)}, \vec{y}^{(2)}, \ldots, \vec{y}^{(p)}]} \right)$

π_n: proj: $\mathbb{R}^n \twoheadrightarrow \mathbb{R}$ and

σ is cyclic shift to the left, ie.,

$$\sigma \overline{[\vec{y}^{(1)}, \vec{y}^{(2)}, \ldots, \vec{y}^{(n)}]} = \overline{[\vec{y}^{(2)}, \vec{y}^{(3)}, \ldots, \vec{y}^{(n)}, \vec{y}^{(1)}]}$$

<u>A Remark</u>:

The above theorem shows that certain periodic n-dimensional c.f. converge.

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we will use following lemma in the proof of the theorem.

Lemma (Power Method)

If M is a Perron-Frobenius (P.F.) matrix with spectral radius λ and x^* is the corresponding eigenvector, then $\forall z \in \mathbb{R}^n$ with $z \geq 0$ we have,

$$\lim_{m\to\infty} M^m z \overset{p}{=} x^*.$$

Proof of the theorem:

Let $A_1 := D[\vec{y}^{(1)}]$ and for $s \geq 2$ inductively

- choose m & $\nu \in \{1, 2, \dots, p\}$ such that
$$s = pm + \nu$$
- Define $A_s := A_{s-1}\left(D[\vec{y}^{(\nu)}]\right)$

we want to prove that

$$\begin{pmatrix} 1 \\ w \end{pmatrix} \overset{p}{=} \lim_{k\to\infty} A_k \begin{pmatrix} 0 \\ e_n \end{pmatrix} \quad \text{— } (*)$$

Let $m \in \mathbb{N}$ and $\nu \in \{1, 2, \ldots, p\}$ be arbitrary.

Inductive def^{n} of A_K

$$\lim_{m\to\infty} A_{pm+\nu}\begin{pmatrix}0\\ e_n\end{pmatrix}$$

$$= \lim_{m\to\infty} A_{pm}\left(A_\nu\begin{pmatrix}0\\ e_n\end{pmatrix}\right) \quad \because M = A_p$$

$$= \lim_{m\to\infty} M^m\left(A_\nu\begin{pmatrix}0\\ e_n\end{pmatrix}\right) \underline{\underline{p}} \begin{pmatrix}1\\ \omega\end{pmatrix} \quad \text{Q.E.D.}$$

power lemma

$$\boxed{\text{Proof of } \vec{\omega} = \left[\overrightarrow{\vec{y}^{(1)}, \vec{y}^{(2)}, \ldots, \vec{y}^{(p)}}\right]}$$

For $i = 1, 2, \cdots, p$ define

$$F_i := C_n\left[\vec{y}^{(i)}\right] \circ C_n\left[\vec{y}^{(i+1)}\right] \circ \cdots \circ C_n\left[\vec{y}^{(p)}\right]$$

Note that if $\omega > 0$, then $C_n[\vec{y}](\omega) > 0$ for every $\vec{y} \in \mathbb{N}_0^n$ and hence $F_i(\omega) > 0$.

Also,

$$D_n\left[\vec{y}^{(p)}\right]\begin{pmatrix}1\\ \omega\end{pmatrix} = \pi_n(\omega)\begin{pmatrix}1\\ F_p(\omega)\end{pmatrix} \text{ and}$$

$$D_n\left[\vec{y}^{(i-1)}\right]\begin{pmatrix}1\\ F_i(\omega)\end{pmatrix} = \pi_n\left(F_i(\omega)\right)\begin{pmatrix}1\\ F_{i-1}(\omega)\end{pmatrix}$$

for every $i = 2, 3, \cdots, p$.

$\because \begin{pmatrix}1\\ \omega\end{pmatrix}$ is eigenvector associated with λ,

$$\lambda\begin{pmatrix}1\\ \omega\end{pmatrix} = M\begin{pmatrix}1\\ \omega\end{pmatrix}$$

$$= D_n\left[\vec{y}^{(1)}\right] \times \cdots \times D_n\left[\vec{y}^{(p)}\right]\begin{pmatrix}1\\ \omega\end{pmatrix}$$

$$= D_n\left[\vec{y}^{(1)}\right] \times \cdots \times D_n\left[\vec{y}^{(p-1)}\right]\pi(\omega)\begin{pmatrix}1\\ F_p(\omega)\end{pmatrix}$$

$\vdots$

$$\vdots$$

$$= \pi(\vartheta) \times \pi(F_p(\vartheta)) \times \cdots \times \pi(F_2(\vartheta)) \begin{pmatrix} 1 \\ F_1(\vartheta) \end{pmatrix} \quad \text{—(A)}$$

So,

$$\vartheta = F_1(\vartheta) \quad \text{and}$$

$$\lambda = \pi(\vartheta) \times \pi(F_p(\vartheta)) \times \cdots \times \pi(F_2(\vartheta))$$

$$= \pi(F_1(\vartheta)) \times \cdots \times \pi(F_p(\vartheta)) \quad \text{— call this eq}^n\ (**)$$

For any $j = 1, 2, \ldots, p$ we have,

$$\begin{pmatrix} 1 \\ F_j(\vartheta) \end{pmatrix} \overset{p}{=} D_n[\vec{y}^{(j)}] \times \cdots \times D_n[\vec{y}^{(p)}] \begin{pmatrix} 1 \\ \vartheta \end{pmatrix}$$

$$= D_n[\vec{y}^{(j)}] \times \cdots \times D_n[\vec{y}^{(p)}] \overline{[\vec{y}^{(1)}, \ldots, \vec{y}^{(p)}]}$$

$$= [\vec{y}^{(j)}, \ldots, \vec{y}^{(p)}, \overline{\vec{y}^{(1)}, \ldots, \vec{y}^{(p)}]}$$

$$= \overline{[\vec{y}^{(j)}, \vec{y}^{(j+1)}, \ldots, \vec{y}^{(p)}]}$$

$$= 6^{j-1}\left[\overline{\vec{y}^{(1)},\dots,\vec{y}^{(p)}}\right]$$

$$\therefore \begin{pmatrix} 1 \\ F_j(\omega) \end{pmatrix} = 6^{j-1}\left[\overline{\vec{y}^{(1)},\dots,\vec{y}^{(p)}}\right]$$

$$\pi_n\big(F_j(\omega)\big) = \pi_n\left(6^{j-1}\left[\overline{\vec{y}^{(1)},\dots,\vec{y}^{(p)}}\right]\right)$$

using above and equation $(**)$ we have that

$$\lambda = \prod_{j=1}^{p} \pi_n\left(6^{j-1}\left[\overline{\vec{y}^{(1)},\dots,\vec{y}^{(p)}}\right]\right)$$

The Theorem is proved.

References:

(I) Bauer, M. (1992). Dilatations & Continued Fractions. Linear Algebra & its Applications. 174. 183–213.

(II) Schweiger, F. Multidimensional Continued Fractions. 2000. Oxford University Press Inc. - New York.