Analysis Qualifying Examination 1

Please start each problem on a new page and remember to write your code on each page of your answers.

You should exercise good judgement in deciding what constitutes an adequate solution. In particular, you should not try to solve a problem by just quoting a theorem that reduces what you are asked to prove to a triviality. If you are not sure whether you may use a particular theorem, ask the proctor.

- **1.** Let *m* be Lebesgue measure on \mathbb{R} and let *A* be a Lebesgue measurable subset of \mathbb{R} such that m(A) > 0.
- [8] (a) Let t < 1. Prove that there is an interval I such that $m(A \cap I) > tm(I)$.
- [8] (b) Prove that there exists $\delta > 0$ such that $(-\delta, \delta) \subseteq A A$, where $A A = \{x y : x, y \in A\}$. (Part (a) should help.)
- [17] **2.** Let (X, \mathcal{M}, μ) be a finite measure space and let $f: X \to [0, \infty]$ be measurable. Prove that $\int f d\mu < \infty$ if and only if $\sum_{n=1}^{\infty} \mu(f > n) < \infty$.

[17] **3.** Find
$$\lim_{n \to \infty} \int_0^\infty \frac{n \sin(x/n)}{x(1+x^2)} dx$$
. Be sure to justify your calculations.

- [17] **4.** Let (X, \mathscr{A}, μ) be a measure space with $\mu(X) < \infty$. Let $f, f_1, f_2, \ldots : X \to \mathbb{R}$ be measurable. Prove that $f_n \to f$ in measure if and only if $\int \frac{|f f_n|}{1 + |f f_n|} d\mu \to 0$.
- [17] 5. Let $f, g: [a, b] \to \mathbb{C}$ be absolutely continuous. Prove that

$$\int_{a}^{b} f'(x)g(x) \, dx = f(b)g(b) - f(a)g(a) - \int_{a}^{b} f(x)g'(x) \, dx.$$

Be sure to explain where and how absolute continuity is used. If your proof contains a step of the form $\int (u+v) = \int u + \int v$, be sure to explain why it is justified. (After all, for instance, if v = -u but u is not integrable, then such a step would not be justified.)

[16] **6.** Let $f: [0,1] \to \mathbb{R}$ be continuous and let $\varepsilon > 0$. Prove that there is a function g on [0,1] of the form

$$g(x) = \sum_{k=0}^{n} c_k x^{4k},$$

where $n \in \mathbb{N}$ and $c_1, \ldots, c_n \in \mathbb{Q}$, such that for each $x \in [0, 1]$, $|f(x) - g(x)| < \varepsilon$.