1. Define $T$ on $\ell^\infty$ by $(Tf)(n) = \frac{1}{n} f(n)$. Clearly $T : \ell^\infty \to \ell^\infty$. Let

$$B = \{ f \in \ell^\infty : \|f\|_\infty \leq 1 \}.$$

Prove that $T[B]$ is a compact subset of $\ell^\infty$.

2. Let $X$ and $Y$ be normed linear spaces, let $T : X \to Y$ be a linear map, let $X_w$ be $X$ with its weak topology, and let $Y_w$ be $Y$ with its weak topology.

   (a) Prove that if $T$ is continuous from $X$ to $Y$, then $T$ is continuous from $X_w$ to $Y_w$.

   (b) Prove that if $T$ is continuous from $X$ to $Y_w$, then $T$ is continuous from $X$ to $Y$.

3. Let $m$ be Lebesgue measure on $[0,1]$. Show that $L^2(m)$ is separable and $L^\infty(m)$ is not separable.

4. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $L^p = L^p(\mu)$ for $1 \leq p \leq \infty$, and let $g \in L^\infty$. Let $p \in [1, \infty)$. Define $T$ on $L^p$ by $Tf = gf$.

   (a) Prove that $T$ is a continuous linear map from $L^p$ into $L^p$.

   (b) Suppose $\mu$ is semifinite. Prove that $\|T\| = \|g\|_\infty$.

5. Let $X$ be a Hausdorff space, let $\mathcal{G}$ be the collection of open subsets of $X$, and let $\mathcal{K}$ be the collection of compact subsets of $X$. Let $\mu$ be a Radon outer measure on $X$. Let $A \subseteq X$ such that $A$ is $\mu$-measurable and $\mu(A) < \infty$. Prove that $\mu(A) = \sup \{ \mu(C) : A \supseteq C \in \mathcal{K} \}$.

6. Let $f \in L^1(\mathbb{T})$. For each $k \in \mathbb{Z}$, let $\hat{f}(k) = \int_0^1 e^{-2\pi i k x} f(x) \, dx$ be the $k$-th Fourier coefficient of $f$. Prove that $\hat{f}(k) \to 0$ as $|k| \to \infty$. In other words, prove the Riemann-Lebesgue lemma for Fourier series.

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1 To say that $\mu$ is semifinite means that for each $B \in \mathcal{A}$, if $\mu(B) = \infty$, then there exists $A \in \mathcal{A}$ such that $A \subseteq B$ and $0 < \mu(A) < \infty$.

2 To say that $\mu$ is a Radon outer measure on $X$ means that (a) $\mu$ is an outer measure on $X$, (b) for each $G \in \mathcal{G}$, $G$ is $\mu$-measurable and $\mu(G) = \sup \{ \mu(K) : G \supseteq K \in \mathcal{K} \}$, (c) for each $E \subseteq X$, $\mu(E) = \inf \{ \mu(G) : E \subseteq G \in \mathcal{G} \}$, and (d) for each $x \in X$, $\mu(\{x\}) < \infty$. 