Analysis Qualifying Examination 2 **Tuesday, August 16, 2022**

Please start each problem on a new page and remember to write your code on each page of your answers.

You should exercise good judgement in deciding what constitutes an adequate solution. In particular, you should not try to solve a problem by just quoting a theorem that reduces what you are asked to prove to a triviality. If you are not sure whether you may use a particular theorem, ask the proctor.

[17] **1.** Define T on ℓ^∞ by $(Tf)(n) = \frac{1}{n} f(n)$. Clearly $T\colon \ell^\infty \to \ell^\infty$. Let

$$B = \{\, f \in \ell^\infty : \|f\|_\infty \leq 1 \,\}.$$

Prove that $T[B]$ is a compact subset of ℓ^∞.

2. Let X and Y be normed linear spaces, let $T\colon X \to Y$ be a linear map, let X_w be X with its weak topology, and let Y_w be Y with its weak topology.

[8] **(a)** Prove that if T is continuous from X to Y, then T is continuous from X_w to Y_w.

[8] **(b)** Prove that if T is continuous from X to Y_w, then T is continuous from X to Y.

[17] **3.** Let m be Lebesgue measure on $[0,1]$. Show that $L^2(m)$ is separable and $L^\infty(m)$ is not separable.

4. Let $(X, \mathscr{A}, \mu)$ be a measure space, let $L^p = L^p(\mu)$ for $1 \leq p \leq \infty$, and let $g \in L^\infty$. Let $p \in [1, \infty)$. Define T on L^p by $Tf = gf$.

[6] **(a)** Prove that T is a continuous linear map from L^p into L^p.

[11] **(b)** Suppose μ is semifinite.[1] Prove that $\|T\| = \|g\|_\infty$.

[16] **5.** Let X be a Hausdorff space, let $\mathscr{G}$ be the collection of open subsets of X, and let $\mathscr{K}$ be the collection of compact subsets of X. Let μ be a Radon outer measure[2] on X. Let $A \subseteq X$ such that A is μ-measurable and $\mu(A) < \infty$. Prove that $\mu(A) = \sup\{\, \mu(C) : A \supseteq C \in \mathscr{K} \,\}$.

[17] **6.** Let $f \in L^1(\mathbb{T})$. For each $k \in \mathbb{Z}$, let $\hat{f}(k) = \int_0^1 e^{-2\pi i k x} f(x)\, dx$ be the k-th Fourier coefficient of f Prove that $\hat{f}(k) \to 0$ as $|k| \to \infty$. In other words, prove the Riemann-Lebesgue lemma for Fourier series.

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[1] To say that μ *is semifinite* means that for each $B \in \mathscr{A}$, if $\mu(B) = \infty$, then there exists $A \in \mathscr{A}$ such that $A \subseteq B$ and $0 < \mu(A) < \infty$.

[2] To say that μ *is a Radon outer measure on* X means that **(a)** μ is an outer measure on X, **(b)** for each $G \in \mathscr{G}$, G is μ-measurable and $\mu(G) = \sup\{\, \mu(K) : G \supseteq K \in \mathscr{K} \,\}$, **(c)** for each $E \subseteq X$, $\mu(E) = \inf\{\, \mu(G) : E \subseteq G \in \mathscr{G} \,\}$, and **(d)** for each $x \in X$, $\mu(\{\, x \,\}) < \infty$.