2023 Gordon examination problems

1. Given 2023 integers $n_1, n_2, \ldots, n_{2023}$, prove that there is a nonempty set $J \subseteq \{1, \ldots, 2023\}$ such that the sum $\sum_{j \in J} n_j$ is divisible by 2023.

2. Find all rational numbers α such that $\cos(\pi\alpha)$ is also rational.

3. If every point of the plane is painted in one of nine colors, do there necessarily exist two points of the same color exactly one inch apart?

4. Let A and B be $n \times n$ real matrices such that $A^2 = A$, $B^2 = B$, and $I - (A + B)$ is invertible. Prove that A and B have the same rank.

5. Let $z_1, \ldots, z_n$ be complex numbers. Prove that there is a nonempty set $J \subseteq \{1, \ldots, n\}$ such that

$$\Big|\sum_{j \in J} z_j\Big| \geq \frac{1}{4\sqrt{2}} \sum_{j=1}^{n} |z_j|.$$

6. Evaluate $\lim_{n \to \infty} n \sin(2\pi e n!)$.