2023 Rasor-Bareis examination problems

1. Subdivide the regular hexagon into 8 congruent quadrilaterals.

2. If every point of the plane is painted in one of three colors, do there necessarily exist two points of the same color that are exactly one inch apart? (You have to justify your answer, of course.)

3. Prove that any triangle ABC whose sides all have length ≤ 1 can be covered by the three discs with centers at A, B, and C and radius $1/\sqrt{3}$.

4. Suppose a polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with nonzero integer coefficients has n distinct integer roots that are pairwise coprime. Prove that the integers a_0 and a_1 are also coprime.

5. Evaluate $\int_0^{2\pi} \lfloor 2023 \sin x \rfloor \, dx$ (where $\lfloor a \rfloor$ denotes the integer part of a, i.e. the maximal intger not exceeding a, so that, for example, $\lfloor \pi \rfloor = 3$ and $\lfloor -1.2 \rfloor = -2$).

6. Determine all functions $f\colon \mathbb{Z} \longrightarrow \mathbb{Z}$ satisfying $f(2x) + 2f(y) = f(f(x+y))$ for all $x, y \in \mathbb{Z}$.