## 2023 Gordon exam solutions

**1.** Given 2023 integers  $n_1, n_2, \ldots, n_{2023}$ , prove that there is a nonempty set  $J \subseteq \{1, \ldots, 2023\}$  such that the sum  $\sum_{i \in J} n_i$  is divisible by 2023.

Solution. Consider the sequence of 2023 sums  $s_1 = n_1$ ,  $s_2 = n_1 + n_2$ , ...,  $s_{2023} = n_1 + n_2 + \cdots + n_{2023}$ . If  $s_k = 0 \mod 2023$  for some k, we are done. If  $s_k \neq 0 \mod 2023$  for all k, then by the pigeonhole principle there are k < l such that  $s_k = s_l \mod 2023$ ; then  $n_{k+1} + \cdots + n_l = s_l - s_k = 0 \mod 2023$ .

## **2.** Find all rational numbers $\alpha$ such that $\cos(\pi \alpha)$ is also rational.

Solution. Let  $\cos(\pi\alpha) \in \mathbb{Q} \setminus \{0\}$ . Consider the sequence  $q_0 = \cos(\pi\alpha)$ ,  $q_1 = \cos(2\pi\alpha)$ ,  $q_2 = \cos(4\pi\alpha)$ , and  $q_k = \cos(2^k\pi\alpha)$ ,  $k \in \mathbb{N}$ . We then have  $q_{k+1} = 2q_k^2 - 1$  for all k, so, by induction,  $q_k \in \mathbb{Q}$  for all k.

If  $\frac{m}{n}$ ,  $m \in \mathbb{Z}$ ,  $n \in \mathbb{N}$ , is the lowest terms representation of a nonzero  $q \in \mathbb{Q}$ , then  $2q^2 - 1 = \frac{2m^2 - n^2}{n^2}$ . If n is odd,  $2m^2 - n^2$  and  $n^2$  are coprime; if n is even, the lowest terms representation of  $2q^2 - 1$  is  $\frac{m^2 - n^2/2}{n^2/2}$ . In any case, the denominator  $n^2$  or  $n^2/2$  of  $2q^2 - 1$  is larger than the denominator n of q, unless n = 1 or 2.

So, the denominators of  $q_k$  increase, unless  $\cos(\pi \alpha) = q_0 = \pm 1, \pm \frac{1}{2}$ , or 0. On the other hand, if  $\alpha \in \mathbb{Q}$ , the sequence  $2^k \pi \alpha$  is eventually periodic modulo  $\pi$ , thus the sequence  $q_k$  must also be eventually periodic. Hence, the only rational  $\alpha$  for which  $\cos(\pi \alpha)$  is also rational are those for which  $\cos(\pi \alpha) = \pm 1, \pm \frac{1}{2}$ , or 0, that is, of the form d/2 and d/3 with  $d \in \mathbb{Z}$ .

## **3.** If every point of the plane is painted in one of nine colors, do there necessarily exist two points of the same color exactly one inch apart?

Solution. The answer is "No". Tile the plane with the  $r \times r$  squares, with r to be specified, and color the squares in the following order:

1	2	3	1	2	3	1	2	3	
4	5	6	4	5	6	4	5	6	
7	8	9	7	8	9	7	8	9	
1	2	3	1	2	3	1	2	3	
4	5	6	4	5	6	4	5	6	

(The points on the boundaries of the squares are colored arbitrarily in the color of one of the adjacent squares.) Then the maximal distance between two points of one square is  $\sqrt{2}r$ , and the minimal distance between two points of two distinct squares of the same color is 2r. So, if  $\sqrt{2}r < 1 < 2r$  there are no two points of the same color at the distance 1.

**4.** Let A and B be  $n \times n$  real matrices such that  $A^2 = A$ ,  $B^2 = B$ , and I - (A + B) is invertible. Prove that A and B have the same rank.

Solution. We have  $A(I - (A + B)) = A - A^2 - AB = AB$  and  $(I - (A + B))B = B - AB - B^2 = AB$ , so  $(I - (A + B))B(I - (A + B))^{-1} = A$ , A and B are conjugate and have the same rank.

Another solution. Let  $C = (I - (A + B))^{-1}$ , then I = C(I - (A + B)) = C - CA - CB. Multiplying it from the right by A we get  $A = CA - CA^2 - CBA = -CBA$ , and similarly, B = -CAB. So, rank  $A = \operatorname{rank}(CBA) \leq \operatorname{rank} B$  and rank  $B = \operatorname{rank}(CAB) \leq \operatorname{rank} A$ , so rank  $A = \operatorname{rank} B$ .

**5.** Let  $z_1, \ldots, z_n$  be complex numbers. Prove that there is a nonempty set  $J \subseteq \{1, \ldots, n\}$  such that

$$\left|\sum_{j\in J} z_j\right| \ge \frac{1}{4\sqrt{2}} \sum_{j=1}^n |z_j|.$$

Solution. Subdivide the complex plane into four segments:  $A_1 = \{z : \operatorname{Re} z \ge |\operatorname{Im} z|\}, A_2 = \{z : \operatorname{Im} z \ge |\operatorname{Re} z|\},$   $A_3 = \{z : \operatorname{Re} z \le -|\operatorname{Im} z|\}, \text{ and } A_4 = \{z : \operatorname{Im} z \le -|\operatorname{Re} z|\}:$ Let  $S = \sum_{j=1}^n |z_j|$ . For some k we have  $\sum_{j:z_j \in A_k} |z_j| \ge S/4.$ W.l.o.g. assume that k = 1. For every  $z \in A_1$  we have  $|z| \le \operatorname{Re} z\sqrt{2}$ , so



$$\sum_{j:z_j \in A_1} |z_j| \le \sum_{j:z_j \in A_1} \sqrt{2} \operatorname{Re} z_j = \sqrt{2} \operatorname{Re} \left( \sum_{j:z_j \in A_1} z_j \right) \le \sqrt{2} \left| \sum_{j:z_j \in A_1} z_j \right|,$$

so  $S \leq 4\sqrt{2} \left| \sum_{j: z_j \in A_1} z_j \right|$ .

6. Evaluate  $\lim_{n \to \infty} n \sin(2\pi e n!)$ .

Solution. Since  $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots$ , for any n we have

$$en! = n! + \frac{n!}{1!} + \frac{n!}{2!} + \dots + \frac{n!}{n!} + \frac{n!}{(n+1)!} + \dots = m_n + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \dots = m_n + x_n$$

where  $m_n \in \mathbb{N}$  and  $\frac{1}{n+1} < x_n < \frac{1}{n+1} + \frac{1}{(n+1)^2} + \dots = \frac{1}{n+1} = \frac{1}{n}$ . We then have  $\sin(2\pi e n!) = \sin(2\pi x_n)$ ; since  $x_n \longrightarrow 0$ ,  $\sin(2\pi x_n)/(2\pi x_n) \longrightarrow 1$  as  $n \longrightarrow \infty$ , so  $\lim_{n \to \infty} n \sin(2\pi e n!) = \lim_{n \to \infty} n(2\pi x_n) = 2\pi$ .