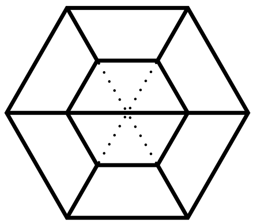
2023 Rasor-Bareis exam solutions

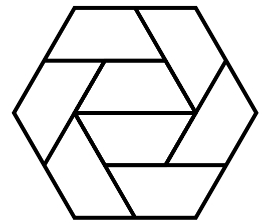
1. *Subdivide the regular hexagon into 8 congruent quadrilaterals.*

Solution.

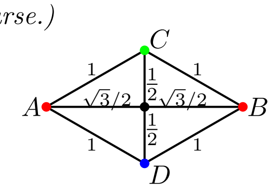


(Assuming the sides of the hexagon have length 1, all eight quadrilaterals are trapezoids with sides $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ and angles $\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}$.)

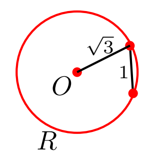
Another solution.



2. *If every point of the plane is painted in one of three colors, do there necessarily exist two points of the same color that are exactly one inch apart? (You have to justify your answer, of course.)*

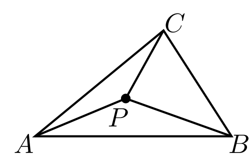


Solution. The answer is "Yes". Suppose the plane is colored red, blue, and green so that no two points at distance 1 have the same color. We then claim that any two points at the distance of $\sqrt{3}$ from each other have same color. Indeed, for any two points A and B with $|AB| = \sqrt{3}$, let C and D be the points on the midperpendicular of the segment AB at the distance $1/2$ from AB. Then $|CD| = |CA| = |DA| = |CB| = |DB| = 1$ so, under our assumption, if, say, A is red, then C and D must be blue and green, and B is also red.



Now choose any point O; w.l.o.g. assume that O is red. Then all points of the circle R centered at O and of radius $\sqrt{3}$ are red. There are two points on R at distance 1 from each other, which leads to contradiciton.

3. *Prove that any triangle ABC whose sides all have length ≤ 1 can be covered by the three discs with centers at A, B, and C and radius $1/\sqrt{3}$.*



Solution. Suppose there is a point P inside the triangle that is not covered by the discs, so that $|AP|, |BP|, |CP| > 1/\sqrt{3}$. At least one of the angles $\angle APB$, $\angle BPC$, $\angle CPA$ is $\geq 2\pi/3$; without loss of generality, assume that $\angle APB \geq 2\pi/3$, so that $\cos(\angle APB) \leq -1/2$. Then by the cosine theorem, $|AB|^2 = |AP|^2 + |BP|^2 - 2|AP|\,|BP|\cos(\angle APB) > 1/3 + 1/3 + 1/3 = 1$.

4. *Suppose a polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with nonzero integer coefficients has n distinct integer roots that are pairwise coprime. Prove that the integers a_0 and a_1 are also coprime.*

Solution. If $n = 1$, $a_1 = 0$ and so, a_0, a_1 are coprime. Let $n \geq 2$. Let $r_1, \ldots, r_n$ be the roots of f, then $a_0 = \pm r_1 \cdots r_n$ and $a_1 = \mp \sum_{i=1}^{n} r_1 \cdots r_{i-1} r_{i+1} \cdots r_n$. Let p be a prime that divides a_0, then p divides r_i for some i, and since the integers $r_1, \ldots, r_n$ are pairwise coprime, there is exactly one such i. Hence, p divides all the summands of a_1 except for $r_1 \cdots r_{i-1} r_{i+1} \cdots r_n$, and so, doesn't divide a_1.

5. *Evaluate $\int_0^{2\pi} \lfloor 2023 \sin x \rfloor \, dx$ (where $\lfloor a \rfloor$ denotes the integer part of a, i.e. the maximal intger not exceeding a, so that, for example, $\lfloor \pi \rfloor = 3$ and $\lfloor -1.2 \rfloor = -2$).*

Solution. For any $y \neq 0$, $\lfloor y \rfloor + \lfloor -y \rfloor = -1$, and for any $x \in (0, \pi)$, $\sin(\pi + x) = -\sin x \neq 0$. So,

$$\int_0^{2\pi} \lfloor 2023 \sin x \rfloor \, dx = \int_0^{\pi} \lfloor 2023 \sin x \rfloor \, dx + \int_{\pi}^{2\pi} \lfloor 2023 \sin x \rfloor \, dx = \int_0^{\pi} \lfloor 2023 \sin x \rfloor + \lfloor -2023 \sin x \rfloor \, dx$$
$$= \int_0^{\pi} -1 \, dx = -\pi.$$

(More exactly, the function under the last integral is $f(x) = \begin{cases} -1, & x \neq 0, \pi \\ 0, & x = 0, \pi \end{cases}$, but $\int_0^{\pi} f = -\pi$ anyway.)

6. *Determine all functions $f \colon \mathbb{Z} \longrightarrow \mathbb{Z}$ satisfying $f(2x) + 2f(y) = f(f(x+y))$ for all $x, y \in \mathbb{Z}$.*

Solution. For any $x, y \in \mathbb{Z}$ we have $f(2(x-1))+2f(y+1) = f(f(x+y)) = f(2x)+2f(y)$, so $2(f(y+1)-f(y)) = f(2x)-f(2x-2)$. Taking $x = 0$ we obtain that for all $y \in \mathbb{Z}$, $f(y+1)-f(y) = c$ where $c = (f(0)-f(-2))/2$. Thus, f is linear, $f(y) = cy + d$ for $d = f(0)$. The equation $f(2x) + 2f(y) = f(f(x + y))$ takes the form $2cx+d+2cy+2d = c(cx+cy+d)+d$, so $2cx+2cy+2d = c^2x+c^2y+cd$ for all x, y, so $c^2 = 2c$ and $cd = 2d$. Hence either $c = d = 0$ or $c = 2$ and d is arbitrary, that is, $f = 0$ or $f(x) = 2x + d$ with $d \in \mathbb{Z}$.