## Algebra Qualifying Exam I

2022

## Instructions

Enter your name. # on the roster sheet and a code name for yourself that is different from any code name that has already been entered.

Answer each question on a separate sheet or sheets of paper, and write your code name and the problem number on each sheet of paper that you submit for grading. Do not put your real name on any sheet of paper that you submit for grading.

Answer as many questions as you can. Do not use theorems which make the solution to the problem trivial. Always clearly display your reasoning. The judgment you use in this respect is an important part of the exam.

This is a closed book, closed notes exam.

## 2022 Algebra 6111 Qualifying Exam

1. Prove that if G is a finite group of order n, and p is the smallest prime dividing n then any subgroup of G of index p is normal.

2. Show that every Sylow *p*-subgroup of a finite nilpotent group is normal.

3. Let  $\mathbb{C}$  denote the field of complex numbers. Show that the ideal in  $\mathbb{C}[x, y]$  generated by  $x + y^2$  and  $y + x^2 + 2xy^2 + y^4$  is a maximal ideal.

4. Let R be a ring and  $_{R}M$  a left R-module. Prove that if  $_{R}M$  is Noetherian and  $f: M \to M$  a surjective R-linear map, then f is an isomorphism.

5. Let  $\mathbb{Z}^{(n)}$  be the free  $\mathbb{Z}$ -module with base  $(e_1, \ldots, e_n)$ . Let K be the submodule generated by the elements  $f_i = \sum_{j=1}^n a_{ij}e_j$  where  $a_{ij} \in \mathbb{Z}$  and  $d = \det(a_{ij}) \neq 0$ . Show that  $|\mathbb{Z}^{(n)}/K| = |d|$ .