Algebra Qualifying Exam II

2022

Instructions

Enter your name.# on the roster sheet together with a code name for yourself that is different from any code name that has already been entered.

Answer each question on a separate sheet or sheets of paper, and write your code name and the problem number on each sheet of paper that you submit for grading. Do not put your real name on any sheet of paper that you submit for grading.

Answer as many questions as you can. Do not use theorems which make the solution to the problem trivial. Always clearly display your reasoning. The judgment you use in this respect is an important part of the exam.

This is a closed book, closed notes exam.

2022 Algebra 6112 Qualifying Exam

1. Prove Yoneda's Lemma: Let $\mathcal{C}$ be a category and let $\mathbf{Func}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})$ be the category of contravariant functors from $\mathcal{C}$ to $\mathbf{Sets}$. Recall that for $X \in \mathrm{Ob}(\mathcal{C})$, we define $h_X := \mathrm{Hom}_{\mathcal{C}}(-, X)$ in $\mathbf{Func}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})$. Prove that for every $F \in \mathbf{Func}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})$ there exists a bijection between $\mathrm{Hom}_{\mathbf{Func}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})}(h_X, F)$ and $F(X)$.

2. Let R be an integral domain with field of fractions K, V be a K-vector space, and I a nonzero ideal in R. Show that all tensors in $I \otimes_R V$ are elementary and that $I \otimes_R V \cong V$ as R-modules.

3. Compute $\mathrm{Ext}_R^{\bullet}(M, N)$, where $R = \mathbb{Z}[x]$, $M = N = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}[x]/(2, x)$.

4. Let E, K and F be subfields of a field L such that $F \subseteq E$ and $F \subseteq K$. Assume that E/F is a finite Galois extension. Prove EK/K is a finite Galois extension and $\mathrm{Gal}(EK/K) \simeq \mathrm{Gal}(E/E \cap K)$.

5. Let F be a field that contains n distinct n^{th} roots of 1. Let E be the splitting field over F of a polynomial of the form

$$f(x) = (x^n - a_1) \cdots (x^n - a_r) \quad \text{with} \quad a_i \in F.$$

Show that E/F is an abelian Galois extension such that $G = \mathrm{Gal}(E/F)$ has exponent m dividing n.