

What is bicycle monodromy?

(July 13, 2023 ; OSU)

Main references :

G. Bor, M. Levi, R. Perline and S. Tabachnikov - Tire tracks and integrable curve equation; IMRN (May 2018).

W.G. Cady - The circular tractrix. AMM (Dec 1965).

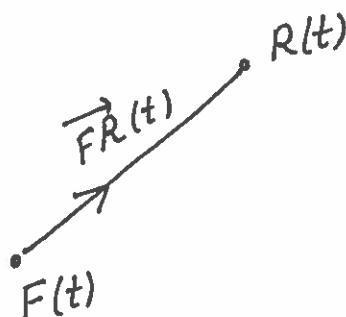
R.L. Foote, M. Levi and S. Tabachnikov - Tractrices, bicycle tire

tracks, hatchet planimeter, and a 100 year old conjecture.
AMM (July 2012).

For animations, see persweb.wabash.edu/facstaff/foote/Planimeter

§1. Given a parametric curve $F(t)$ in \mathbb{R}^2 and $l \in \mathbb{R}_{>0}$,
the tractrix of F (or rear track of a bicycle of length l ,
whose front track is F) is the parametric curve $R(t)$,
uniquely determined by the following constraints, and
the prescription of an initial position.

$$(*) \begin{cases} |\vec{FR}(t)| = l, \quad \forall t \\ \vec{FR}(t) \text{ is tangent to } R(t) \end{cases}$$



Initial condition : $R(0) \in S_l^1(F(0))$
circle of radius l , centered at $F(0)$.

(2)

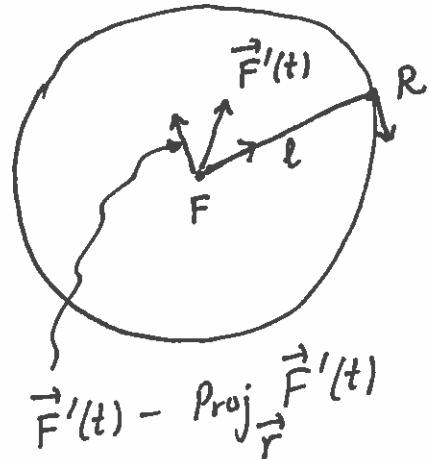
Let $\vec{r}(t)$ = the unit vector along $\vec{FR}(t)$, so that

$$\vec{FR}(t) = l \vec{r}(t); \quad R(t) = F(t) + l \vec{r}(t)$$

Substituting these expressions in (*) gives :

$$l \vec{r}'(t) = -\vec{F}'(t) + (\vec{F}'(t) \cdot \vec{r}(t)) \vec{r}(t) \quad - (B)$$

[Picture explaining the meaning of equation (B)]



Example . Linear tractrix (Newton 1676).

$$F(t) = (t, 0)$$

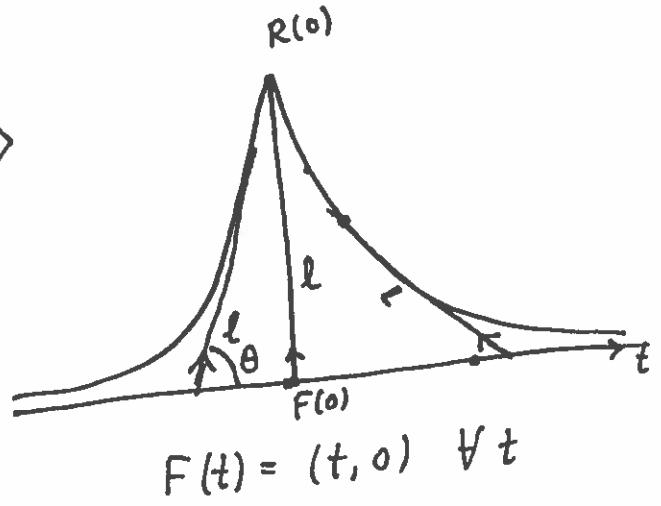
$$\vec{F}'(t) = \langle 1, 0 \rangle$$

$$\text{Let } \vec{r}(t) = \langle \cos(\theta(t)), \sin(\theta(t)) \rangle$$

Initial condition

$$\theta(0) = \theta_0 \in (0, \pi)$$

(If $\theta_0 = 0$ or π , the solution
is $\theta(t) \equiv \theta_0$.)



$$F(t) = (t, 0) \quad \forall t$$

(B) becomes $\theta'(t) = \frac{\sin(\theta(t))}{l}$. Set $p(t) = \tan\left(\frac{\theta(t)}{2}\right)$.

We get $p'(t) = \frac{p(t)}{t}$, solved as $\boxed{p(t) = K \cdot e^{\frac{t}{t}}}$ (3) ($K \in \mathbb{R}$ constant)

$$K = \tan\left(\frac{\theta_0}{2}\right).$$

Remark (on change of variables) $\vec{r}(t) = \langle \cos(\theta), \sin(\theta) \rangle$

in terms of $p = \tan\left(\frac{\theta}{2}\right)$ takes the following form
 ("projective" parametrization of S^1)

$$p \in \mathbb{R} \mapsto \left(\frac{1-p^2}{1+p^2}, \frac{2p}{1+p^2} \right) \in S^1$$

or,

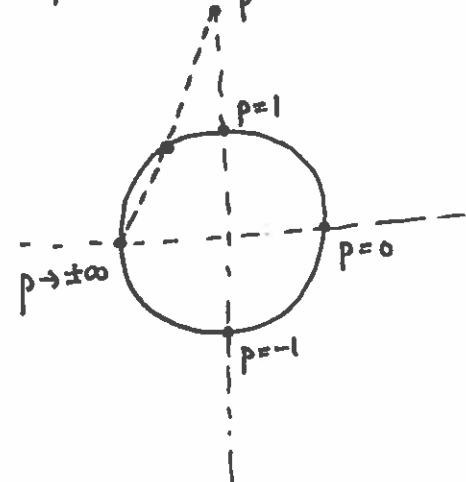
$$\mathbb{R}^2 \setminus \{(0,0)\} \rightarrow S^1$$

$$(a,b) \mapsto \left(\frac{a^2-b^2}{a^2+b^2}, \frac{2ab}{a^2+b^2} \right)$$

gives an identification of

$$\mathbb{P}^1(\mathbb{R}) \xrightarrow{\sim} S^1 \text{ where}$$

$$\mathbb{P}^1(\mathbb{R}) := \mathbb{R}^2 \setminus \{(0,0)\} / \mathbb{R}_{\neq 0} \text{ (scaling action)}$$



Stereographic projection
 parametrizing S^1 .

Ex. Area under the linear tractrix = $\frac{\pi}{2} l^2$.

§2. Bicycle monodromy map.-

Assume $F(t)$ ($0 \leq t \leq T$) is a simple, closed, piecewise smooth curve [piecewise C^1 is usually enough].

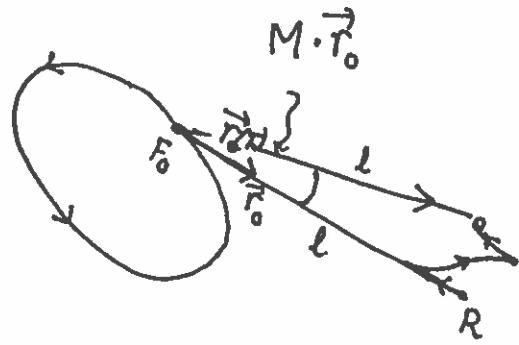
The bicycle monodromy map $M : S^1 \rightarrow S^1$ (or; M_l or $M_{F;l}$ if dependence on F and l needs to be explicitly stated) is given by:

$$M \cdot \vec{r}_0 := \vec{r}(T)$$

where $\vec{r}(t)$ is the

unique solution to equation (B): $l \vec{r}' = -\vec{F}' + \text{Proj}_{\vec{r}} \vec{F}'$ with $\vec{r}(0) = \vec{r}_0$.

In simple terms, $M \cdot \vec{r}_0$ is the final position of the rear wheel after the bicycle front wheel traverses along F , starting with the initial position given by \vec{r}_0 .



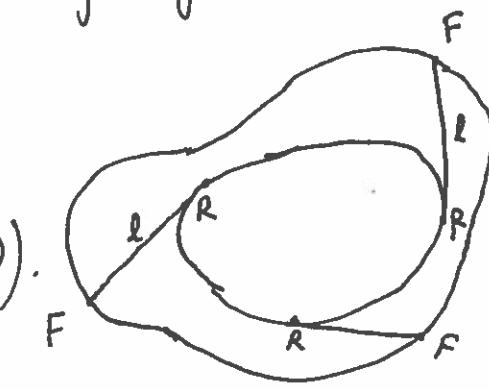
Remark. - (1) Saying \vec{r}_0 is a fixed point of $M_{F;l}$ is same as

saying that the rear wheel trajectory also closes.

(2) For fixed F , $M_{F;l}$ depends

analytically on l , &

$\lim_{l \rightarrow \infty} M_{F;l} = \text{Id}$ (see Thm (3.1) below).



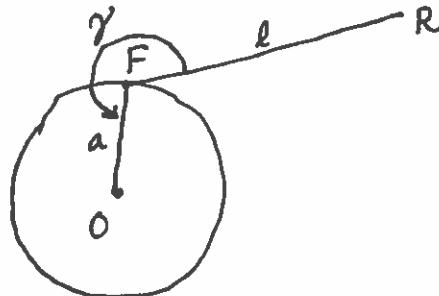
Example - Circular tractrix - Euler 1788.

The front tire moves along circle of radius $a \in \mathbb{R}_{>0}$, centered at O .
 $F(t) = (a \cos(t), a \sin(t))$

Let $\gamma(t) = \text{angle } R(t) F(t) O$

Then eqⁿ (B) becomes

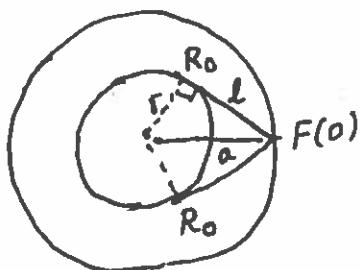
$$\boxed{\gamma'(t) = 1 - \frac{a}{l} \cos(\gamma)}$$



For explicit soln. - see [Cady]

Nature of the monodromy map depends on whether $a < l$; $a = l$ or $a > l$.

- $a < l$. Monodromy $M_l : S^1 \rightarrow S^1$ has no fixed points.
- $a = l$. M_l has exactly one fixed point.
- $a > l$. M_l has two fixed points.



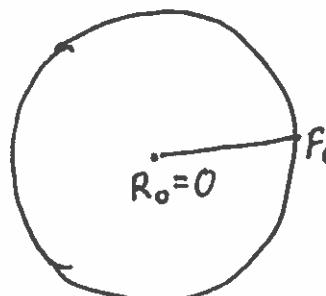
$$a > l$$

$$r = \sqrt{a^2 - l^2}$$

Two possible initial conditions

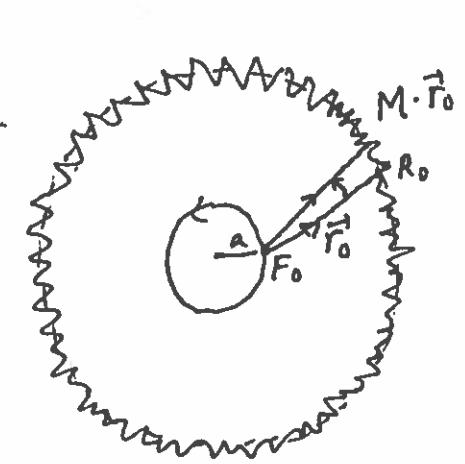
$$\text{with } M_l \vec{r}_0 = \vec{r}_0$$

"hyperbolic case"



$$a = l$$

One fixed point



M is "infinitesimal rotation"

0 fixed points.

"parabolic case"

"elliptic case"

Menzin's conjecture (1906). Proved for Convex regions in [FLT].

(6)

$A_F > \pi l^2 \Rightarrow M_{F;l}$ has a fixed point; which is
 (i.e. we are in
 $A_F = \text{area enclosed by } F$) attracting (i.e. we are in
 hyperbolic case)

§3. Monodromy map and area

$F(t)$: simple, closed, piecewise smooth curve in \mathbb{R}^2
 $0 \leq t \leq T$
 $l \in \mathbb{R}_{>0}$

$\vec{r}(t)$ solves

$$\rightsquigarrow M_{F;l} : S^1 \rightarrow S^1$$

$$M_{F;l} \vec{r}_0 = \vec{r}(T) ; \text{ where}$$

$$\begin{cases} l \vec{r}'(t) = -\vec{F}'(t) + (\vec{F}'(t) \cdot \vec{r}(t)) \vec{r}(t) \\ \vec{r}(0) = \vec{r}_0 \end{cases}$$

Theorem (3.1) . - (Hill 1894) [BLPT]

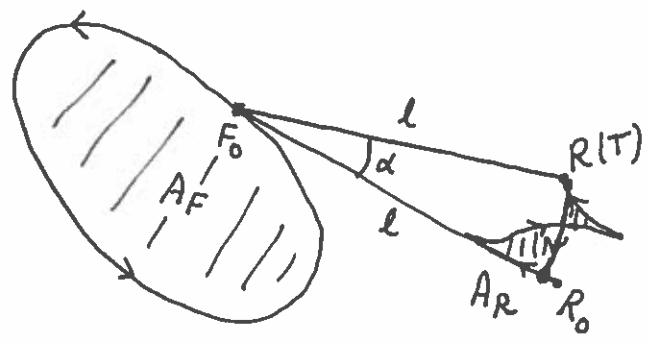
$$M_l \cdot \vec{r}_0 = \vec{r}_0 + \frac{1}{l^2} A_F \cdot \vec{r}_0 + O\left(\frac{1}{l^3}\right) ; \text{ where}$$

$$A_F = \begin{bmatrix} 0 & -A_F \\ A_F & 0 \end{bmatrix}$$

Theorem (3.2) [FLT] . - (a variant due to Prytz 1890)

$$A_F = \alpha l^2 + A_R \text{ where}$$

$A_R = (\text{signed}) \text{ area enclosed by } R(t) \text{ followed by a rotation to bring } R(T) \text{ at } R(0)$



These theorems are the basis of Prytz' planimeter (1886) : (7)

$$\text{i.e., for } l \gg 0, \quad A_F \approx \alpha l^2.$$

Proof of Theorem (3.1) : Let $\varepsilon = \frac{1}{l}$ and write (B) as:

$$\vec{r}'(t) = \varepsilon (-\vec{F}'(t) + (\vec{F}'(t) \cdot \vec{r}(t)) \vec{r}(t))$$

$$\vec{r}(0) = \vec{r}_0.$$

$$\text{Write } \vec{r}(t) = \sum_{n=0}^{\infty} \varepsilon^n \vec{r}(t; n); \quad \begin{aligned} \vec{r}(0; 0) &= \vec{r}_0 \\ \vec{r}(0; n) &= \vec{0} \quad \forall n \geq 1. \end{aligned}$$

The diff'l eqⁿ becomes:

$$\vec{r}'(t; 0) = \vec{0} \quad (\text{so, } \vec{r}(t; 0) = \vec{r}_0 \quad \forall t)$$

$$\vec{r}'(t; 1) = -\vec{F}'(t) + (\vec{F}'(t) \cdot \vec{r}(t; 0)) \vec{r}(t; 0)$$

$$\text{For } n \geq 2 \quad \vec{r}'(t; n) = \sum_{a+b=n-1} (\vec{F}'(t) \cdot \vec{r}(t; a)) \vec{r}(t; b)$$

$$M \cdot \vec{r}_0 = \sum_{n=0}^{\infty} \varepsilon^n \vec{r}(T; n). \quad \text{So, } \vec{r}(T; 0) = \vec{r}_0.$$

$$\begin{aligned} \vec{r}(T; 1) &= \int_0^T \vec{r}'(t; 1) dt = - \int_0^T \vec{F}'(t) dt + \int_0^T (\vec{F}'(t) \cdot \vec{r}_0) \vec{r}_0 dt \\ &= \left[-\vec{F}(t) + (\vec{F}(t) \cdot \vec{r}_0) \vec{r}_0 \right]_0^T = \vec{0}. \end{aligned}$$

$$\text{Note: } \vec{r}(t; 1) = -\vec{F}(t) + (\vec{F}(t) \cdot \vec{r}_0) \vec{r}_0 + \vec{c}.$$

$$\vec{r}(T; 2) = \int_0^T \vec{r}'(t; 2) dt \quad (8)$$

$$= \int_0^T (\vec{F}'(t) \cdot \vec{r}_0) (-\vec{F}(t) + \underline{(\vec{F}(t) \cdot \vec{r}_0)} \vec{r}_0 + \vec{c}) \\ + \underline{(\vec{F}'(t) \cdot (-\vec{F}(t) + (\vec{F}(t) \cdot \vec{r}_0) \vec{r}_0 + \vec{c}))} \vec{r}_0$$

(underlined terms give 0 due to periodicity of F)

$$= - \int_0^T (\vec{F}'(t) \cdot \vec{r}_0) \vec{F}(t) dt = \int_0^T (\vec{F}(t) \cdot \vec{r}_0) \vec{F}'(t) dt.$$

$$= \begin{bmatrix} 0 & -A_F \\ A_F & 0 \end{bmatrix} \vec{r}_0 \quad \left(\begin{array}{l} \text{if } \vec{r}_0 = \begin{bmatrix} a \\ b \end{bmatrix}, \text{ the } 1^{\text{st}} \text{ component of} \\ \vec{r}(T, 2) = \int_0^T (ax(t) + by(t)) x'(t) dt \\ = b \cdot \int_0^T y dx = -A_F \cdot b \end{array} \right)$$

Stokes' theorem / Green's thm.

□

Proof of Theorem (3.2). - Let (x, y) be the coordinates of R .
 $(X = x - l \cos \theta; Y = y - l \sin \theta)$ \overline{F} .

Then

$$X dy - Y dX = x dy - y dx + l \cdot d(-x \sin(\theta) + y \cos(\theta)) \\ + 2l (\sin(\theta) dx - \cos(\theta) dy) + l^2 d\theta$$

Integrate over the trajectory $F(t), R(t)$ joined with $F(t) = F(T)$
 $0 \leq t \leq T$ $R(t) = l \cdot e^{-i(t-T)}$
 $T \leq t \leq T + \alpha$

(9)

$$2A_F = 2A_R + 2l \int \sin(\theta) dx - \cos(\theta) dy$$

[integral is over $(l\cos(\theta), l\sin(\theta))$]

$$\theta = 0 \text{ to } -\alpha$$

□

$$\Rightarrow A_F = A_R + l^2 \cdot \alpha.$$

§4. Bicycle monodromy map is a Möbius transformation
 (Foote, 1998).

In particular, the behaviour of fixed points, as for circular tractrix, persists for any F .

Idea of the proof.- Let us write $\vec{F}'(t) = \langle v_1(t), v_2(t) \rangle$
 $\vec{r} = \langle \cos(\theta), \sin(\theta) \rangle$

$$\mathbb{R}^2 \setminus \{(0,0)\} \xrightarrow{\quad} \mathbb{R}^2 \setminus \{(0,0)\} / \mathbb{R}_{\neq 0} \simeq S^1$$

In projective coord. (a, b)
 (see page 3)

Eqⁿ (B) :

$$l\theta' = v_1 \sin(\theta) - v_2 \cos(\theta)$$

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = \frac{-1}{2l} \begin{bmatrix} v_1 & v_2 \\ v_2 & -v_1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

in $sl_2(\mathbb{R})$
 (i.e. has trace 0)

$$p' = \frac{1}{2l} (-v_2 + 2v_1 p + v_2 p^2)$$

⇒ M is valued in corresponding "Lie group" $PSL_2(\mathbb{R})$
 precisely the group of Möbius transformations. □