2024 Gordon examination problems

1. Suppose that $n = 111\ldots11$ is an integer divisible by 7; prove that n is divisible by 13 as well.

2. The Fibonacci sequence is defined recursively by $F_0 = 1$, $F_1 = 1$, $F_2 = 2$, and $F_{n+2} = F_n + F_{n+1}$ for $n = 1, 2, \ldots$. Prove that for every n, $\sqrt[n]{F_{n+1}} \geq 1 + 1/\sqrt[n]{F_n}$.

3. Suppose that complex numbers $z_1, \ldots, z_5$ satisfy $|z_i| = 1$ for all i and $\sum_{i=1}^{5} z_i = \sum_{i=1}^{5} z_i^2 = 0$. Prove that $z_1, \ldots, z_5$ are the vertices of a regular pentagon.

4. Suppose that all the vertices of an n-gon P in the Euclidean plane have integer coordinates and that the length of all sides of P are also integral. Prove that the perimeter of P is an even integer.

5. Prove that the square of the area of a triangle in $\mathbb{R}^n$ is equal to the sum of the squares of the areas of its projections to the $\binom{n}{2}$ two-dimensional coordinate planes in $\mathbb{R}^n$.

6. Prove that for any two $n \times n$ complex matrices A and B, the characteristic polynomials of AB and BA are equal.