2024 Rasor-Bareis examination problems

1. Let $a_1, \ldots, a_{1013}$ be positive integers not exceeding 2024. Prove that $a_i \mid a_j$ for some $i \neq j$.

2. For a real number $x$, let $\lfloor x \rfloor$ be the integer part of $x$ (the largest integer not exceeding $x$). Define the function $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin \lfloor x \rfloor$; is $f$ periodic?

3. Solve the equation $\sqrt{x + \sqrt{4x + \sqrt{16x + \cdots + \sqrt{4^n x + 3}}} - \sqrt{x}} = 1$.

4. Suppose that a polynomial $f$ with real coefficients has degree $n$ and $n$ distinct real roots $a_1, \ldots, a_n$. Let $b_1, \ldots, b_{n-1}$ be the roots of the derivative $f'$. Prove that $\sum_{i=1}^{n} \sum_{j=1}^{n-1} \frac{1}{a_i - b_j} = 0$.

5. Suppose 2024 points are given in the plane with the property that every triangle formed from any three of those 2024 points has area $\leq 1$. Prove that all of these points lie in a triangle of area $\leq 4$.

6. Let $T$ be a triangle in $\mathbb{R}^3$ and let $T_{x,y}$, $T_{y,z}$, $T_{x,z}$ be the projections of $T$ onto the three coordinate planes of $\mathbb{R}^3$. Prove that $\text{area}(T)^2 = \text{area}(T_{x,y})^2 + \text{area}(T_{y,z})^2 + \text{area}(T_{x,z})^2$. 