2024 Rasor-Bareis examination problems

1. Let $a_1, \ldots, a_{1013}$ be positive integers not exceeding 2024. Prove that $a_i \mid a_j$ for some $i \neq j$.

2. For a real number x, let $\lfloor x \rfloor$ be the integer part of x (the largest integer not exceeding x). Define the function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = \sin\lfloor x \rfloor$; is f periodic?

3. Solve the equation $\sqrt{x + \sqrt{4x + \sqrt{16x + \cdots + \sqrt{4^n x + 3}}}} - \sqrt{x} = 1$.

4. Suppose that a polynomial f with real coefficients has degree n and n distinct real roots $a_1, \ldots, a_n$. Let $b_1, \ldots, b_{n-1}$ be the roots of the derivative f'. Prove that $\sum_{i=1}^{n} \sum_{j=1}^{n-1} \frac{1}{a_i - b_j} = 0$.

5. Suppose 2024 points are given in the plane with the property that every triangle formed from any three of those 2024 points has area ≤ 1. Prove that all of these points lie in a triangle of area ≤ 4.

6. Let T be a triangle in $\mathbb{R}^3$ and let $T_{x,y}$, $T_{y,z}$, $T_{x,z}$ be the projections of T onto the three coordinate planes of $\mathbb{R}^3$. Prove that $\operatorname{area}(T)^2 = \operatorname{area}(T_{x,y})^2 + \operatorname{area}(T_{y,z})^2 + \operatorname{area}(T_{x,z})^2$.