Solutions to 2024 Gordon examination problems

1. *Suppose that $n = 111\dots11$ is an integer divisible by 7; prove that n is divisible by 13 as well.*

Solution. Let the number of 1s in n be k, then $n = (10^k - 1)/9$, so n is divisible by 7 iff $10^k \equiv 1 \bmod 7$ and by 13 iff $10^k \equiv 1 \bmod 13$. Under multiplication modulo 7, 10 has order 6: $10 \equiv 3, 10^2 \equiv 2, \dots, 10^5 = 5 \not\equiv 1 \bmod 7$, $10^6 \equiv 1 \bmod 7$, so n is divisile by 7 iff k is divisible by 6. But $10^6 \equiv 1 \bmod 13$ as well, so when n is divisible by 7 then n is divisible by 13.

Another solution. Let $n_1 = 1$, $n_2 = 11$, $n_3 = 111$, etc. n_k is not divisible by 7 for $1 \le k \le 5$ and is divisible by 7 for $k = 6$, $n_6 = 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$. Since for any k, $n_k = n_6(10^{k-6} + 10^{k-12} + \dots + 10^r) + n_r$ where $r \in \{0, \dots, 5\}$ is the residue of k modulo 6, it follows that n_k is divisible by 7 iff $r = 0$, that is, iff k is divisible by 6. But in this case $n_6 \mid n_k$, and since n_6 is divisible by 13, n_k is divisible by 13 too.

2. *The Fibonacci sequence is defined recursively by $F_0 = 1$, $F_1 = 1$, $F_2 = 2$, and $F_{n+2} = F_n + F_{n+1}$ for $n = 1, 2, \dots$. Prove that for every n, $\sqrt[n]{F_{n+1}} \ge 1 + 1/\sqrt[n]{F_n}$.*

Solution. By the (generalized) arithmetic-geometric mean (AGM) inequality,

$$\sqrt[n]{\tfrac{1}{F_{n+1}}} = \sqrt[n]{\tfrac{F_1}{F_2} \cdot \tfrac{F_2}{F_3} \cdots \tfrac{F_n}{F_{n+1}}} \le \tfrac{1}{n}\Big(\tfrac{F_1}{F_2} + \tfrac{F_2}{F_3} + \dots + \tfrac{F_n}{F_{n+1}}\Big) = \tfrac{1}{n}\Big(\tfrac{F_2 - F_0}{F_2} + \tfrac{F_3 - F_1}{F_3} + \dots + \tfrac{F_{n+1} - F_{n-1}}{F_{n+1}}\Big)$$
$$= \tfrac{1}{n}\Big(\big(1 - \tfrac{F_0}{F_2}\big) + \big(1 - \tfrac{F_1}{F_3}\big) + \dots + \big(1 - \tfrac{F_{n-1}}{F_{n+1}}\big)\Big) = 1 - \tfrac{1}{n}\Big(\tfrac{F_0}{F_2} + \tfrac{F_1}{F_3} + \dots + \tfrac{F_{n-1}}{F_{n+1}}\Big)$$

so, $1 - \sqrt[n]{\frac{1}{F_{n+1}}} \ge \frac{1}{n}\big(\frac{F_0}{F_2} + \frac{F_1}{F_3} + \dots + \frac{F_{n-1}}{F_{n+1}}\big)$. Applying the AGM inequality again, we obtain that $1 - \sqrt[n]{\frac{1}{F_{n+1}}} \ge \sqrt[n]{\frac{F_0}{F_2} \cdot \frac{F_1}{F_3} \cdots \frac{F_{n-1}}{F_{n+1}}} = \sqrt[n]{\frac{1}{F_n F_{n+1}}}$, so $\sqrt[n]{F_{n+1}} \ge 1 + 1/\sqrt[n]{F_n}$.

3. *Suppose that complex numbers $z_1, \dots, z_5$ satisfy $|z_i| = 1$ for all i and $\sum_{i=1}^5 z_i = \sum_{i=1}^5 z_i^2 = 0$. Prove that $z_1, \dots, z_5$ are the vertices of a regular pentagon.*

Solution. Let $p(z) = \prod_{i=1}^5 (z - z_i) = z^5 + a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0$ be the polynomial having $z_1, \dots, z_5$ as its roots. Then $a_4 = -\sum_{i=1}^5 z_i = 0$ and $a_3 = \sum_{1 \le i < j \le 5} z_i z_j = \frac{1}{2}\big(\big(\sum_{i=1}^5 z_i\big)^2 - \sum_{i=1}^5 z_i^2\big) = 0$. The polynomial $z^5 p(1/z) = 1 + a_4 z + a_3 z^2 + a_2 z^3 + a_1 z^4 + a_0 z^5$ has roots $z_1^{-1}, \dots, z_5^{-1}$, the polynomial $\overline{p}(z) = z^5 + \overline{a}_4 z^4 + \overline{a}_3 z^3 + \overline{a}_2 z^2 + \overline{a}_1 z + \overline{a}_0$ has roots $\overline{z}_1, \dots, \overline{z}_5$; since for any i we have $z_i^{-1} = \overline{z}_i$, these two polynomials are equal up to scaling, so $a_1 = \overline{a}_4 = 0$ and $a_2 = \overline{a}_3 = 0$ as well. Hence, $p(z) = z^5 + a_0$. Let α be one of the roots of p; then the five roots of p are $\alpha, \alpha\omega, \alpha\omega^2, \alpha\omega^3, \alpha\omega^4$, where $\omega = e^{2\pi i/5}$, and are the vertices of a regular 5-gon.

4. *Suppose that all the vertices of an n-gon P in the Euclidean plane have integer coordinates and that the length of all sides of P are also integral. Prove that the perimeter of P is an even integer.*

Solution. Let the vertices of P be $(a_i, b_i) \in \mathbb{Z}^2$, $i = 1, \dots, n$; to simplify notation let's also put $(a_{n+1}, b_{n+1}) = (a_1, b_1)$. For every i the length of the ith side of P is $l_i = \sqrt{(a_{i+1} - a_i)^2 + (b_{i+1} - b_i)^2}$ and is given to be integral; thus $l_i \equiv (a_{i+1} - a_i) + (b_{i+1} - b_i) \bmod 2$. Hence, modulo 2, the perimeter of P is

$$\sum_{i=1}^n l_i = \sum_{i=1}^n \Big((a_{i+1} - a_i) + (b_{i+1} - b_i)\Big) = (a_{n+1} - a_1) + (b_{n+1} - b_1) = 0.$$

5. *Prove that the square of the area of a triangle in $\mathbb{R}^n$ is equal to the sum of the squares of the areas of its projections to the $\binom{n}{2}$ two-dimensional coordinate planes in $\mathbb{R}^n$.*

Solution. Let A, B, and C be the vetrices of the triangle, let $u = \overrightarrow{AB} = (a_1, \dots, a_n)$ and $v = \overrightarrow{AC} = (b_1, \dots, b_n)$. The area S of the triangle equals $\frac{1}{2}|u| \cdot |v| \sin\theta$, where $\theta = \angle BAC$, so

$$4S^2 = |u|^2 |v|^2 \sin^2\theta = (u \cdot u)(v \cdot v)(1 - \cos^2\theta) = (u \cdot u)(v \cdot v) - (u \cdot v)^2.$$

Thus

$$4S^2 = \big(\textstyle\sum_{i=1}^n a_i^2\big)\big(\sum_{i=1}^n b_i^2\big) - \big(\sum_{i=1}^n a_i b_i\big)^2 = \sum_{1 \le i < j \le n}(a_i^2 b_j^2 + a_j^2 b_i^2 - 2a_i b_j a_j b_i) = \sum_{1 \le i < j \le n}(a_i b_j - a_j b_i)^2.$$

And, for any $i \ne j$, the projection of $\triangle ABC$ onto the (i, j)th coordinate plane has sides (a_i, b_i), (a_j, b_j) and area $\frac{1}{2}|a_i b_j - a_j b_i|$.

Another solution. Let A, B, and C be the vetrices of the triangle, let P be the parallelogram formed by the vectors $u = \overrightarrow{AB}$ and $v = \overrightarrow{AC}$. The area of P (which is twice the area of $\triangle ABC$) equals the length of $u \wedge v$, an element of the $\binom{n}{2}$-dimensional space $\mathbb{R}^n \wedge \mathbb{R}^n$. (In more details: The 4-linear form $F\colon (\mathbb{R}^n)^4 \longrightarrow \mathbb{R}$ defined by $F(u_1, v_1, u_2, v_2) = (u_1 \cdot u_2)(v_1 \cdot v_2) - (u_1 \cdot v_2)(v_1 \cdot u_2)$ vanishes when $u_1 = v_1$ or $u_2 = v_2$, so induces a bilinear form (an inner product) $B\colon (\mathbb{R}^n \wedge \mathbb{R}^n)^2 \longrightarrow \mathbb{R}$. The corresponding quadratic form $Q(u \wedge v) = B(u \wedge v, u \wedge v) = |u|^2|v|^2 - (u \cdot v)^2$ is the square of the area of the parallelogram formed by u and v, and the length $Q(u \wedge v)^{1/2}$ of $u \wedge v$ with respect to Q equals area(P).) The basis $\{e_i \wedge e_j,\ 1 \le i < j \le n\}$ is orthonormal in $\mathbb{R}^n \wedge \mathbb{R}^n$ with respect to B, so $|u \wedge v|^2 = \sum_{1 \le i < j \le n} s_{i,j}^2$ where $s_{i,j}$ are the coordinates of $u \wedge v$ with respect to this basis. For every $i < j$, $s_{i,j}^2 = Q\big(T_{i,j}(u) \wedge T_{i,j}(v)\big)$ where $T_{i,j}$ is the projection to the (i,j)th coordinate plane, so, $s_{i,j}^2 = \text{area}(T_{i,j}(P))^2$.

6. *Prove that for any two $n \times n$ complex matrices A and B, the characteristic polynomials of AB and BA are equal.*

Solution. If A is invertible, then AB and BA are similar, $BA = A^{-1}(AB)A$, and their characteristic polynomials coincide. The set U of nondegenerate matrices A (that is, with $\det A \neq 0$) is open and dense in the n^2-dimensional $\mathbb{C}$-vector space of the entries of A, and the coefficients of the characteristic polynomials of AB and BA are polynomial functions on this space; since they coincide on U, they coincide everywhere.

Another solution. For variables x, y we have $x(A - yI) - (A - yI)B(A - yI) = (A - yI)(x - B(A - yI)) = (x - (A - yI)B)(A - yI)$, so $\det(A - yI)\det(xI - B(A - yI)) = \det(xI - (A - yI)B)\det(A - yI)$. Since $\det(A - yI)$ is a nonzero polynomial, this implies that $\det(xI - B(A - yI)) = \det(xI - (A - yI)B)$, and putting $y = 0$ we get that $\det(xI - BA) = \det(xI - AB)$.

Yet another solution. Consider the "generic" $n \times n$ matrices $X = (x_{i,j})$ and $Y = (y_{i,j})$ whose $2n^2$ entries are independent commuting variables. Over the field $K = \mathbb{Q}(x_{i,j}, y_{i,j})_{i,j=1}^n$, X and Y are invertible, and thus XY and YX have the same characteristic polynomial $f(t) \in \mathbb{Z}[x_{i,j}, y_{i,j}]_{i,j=1}^n[t]$. Now, given a ring R with elements $a_{i,j}, b_{i,j} \in R$, $i, j = 1, \dots, n$, replacing in f all $x_{i,j}$ by $a_{i,j}$ and $y_{i,j}$ by $b_{i,j}$ we obtain that the matrices $A = (a_{i,j})$ and $B = (b_{i,j})$ have equal characteristic polynomials as well.