Solutions to 2024 Rasor-Bareis examination problems

1. *Let $a_1,\dots,a_{1013}$ be positive integers not exceeding 2024. Prove that $a_i \mid a_j$ for some $i\neq j$.*

Solution. Let $n=1012$, so that $2n=2024$. For every $i=1,\dots,1013$, let $a_i=2^{k_i}b_i$ where b_i is an odd integer. Then $1\le b_i\le a_i\le 2n$; there are only n such odd integers, so $b_i=b_j$ for some $i\neq j$, and then, assuming w.l.o.g. that $k_j\ge k_i$, we have $a_i \mid a_j$.

2. *For a real number x, let $\lfloor x\rfloor$ be the integer part of x (the largest integer not exceeding x). Define the function $f\colon\mathbb{R}\longrightarrow\mathbb{R}$ by $f(x)=\sin\lfloor x\rfloor$; is f periodic?*

Solution. No, f is non-periodic. f takes value 0 on the interval $[0,1)$ only. Indeed, $\sin\lfloor x\rfloor=0$ only if $\lfloor x\rfloor=k\pi$ for some integer k, which is impossible for nonzero k since $\lfloor x\rfloor$ is an integer and $k\pi$ is irrational. Hence, $f(x)=0$ only if $\lfloor x\rfloor=0$. But were f periodic, there would be some $T>0$ such that $f(nT)=0$ for all integer n, and taking $n>1/T$ we get a contradiction.

3. *Solve the equation $\sqrt{x+\sqrt{4x+\sqrt{16x+\cdots+\sqrt{4^nx+3}}}}-\sqrt{x}=1$.*

Solution. For $n=0$, the equation is $\sqrt{x+3}=1+\sqrt{x}$, so $x+3=1+2\sqrt{x}+x$, and the only solution is $x_0=1$. If for some n the equation has a unique solution x_n, then for $n+1$,

$$\sqrt{x+\sqrt{4x+\sqrt{16x+\cdots+\sqrt{4^{n+1}x+3}}}}=1+\sqrt{x} \qquad (*)$$

implies

$$x+\sqrt{4x+\sqrt{16x+\cdots+\sqrt{4^{n+1}x+3}}}=1+2\sqrt{x}+x,$$

so

$$\sqrt{4x+\sqrt{16x+\cdots+\sqrt{4^{n+1}x+3}}}=1+\sqrt{4x},$$

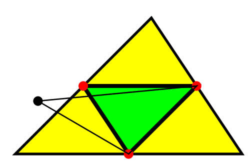
so $4x=x_n$, and the only solution of $(*)$ is $x_{n+1}=x_n/4$. By induction, for every n, the equation has unique solution $x_n=1/4^n$.

4. *Suppose that a polynomial f with real coefficients has degree n and n distinct real roots $a_1,\dots,a_n$. Let $b_1,\dots,b_{n-1}$ be the roots of the derivative f'. Prove that $\sum_{i=1}^n\sum_{j=1}^{n-1}\frac{1}{a_i-b_j}=0$.*

Solution. Since f has no multiple roots, $b_j\neq a_i$ for all i,j. We have $f(x)=c(x-a_1)\cdots(x-a_n)$ for some c, so $\log f(x)=\log c+\sum_{i=1}^n\log(x-a_i)$ and $f'(x)/f(x)=(\log f(x))'=\sum_{i=1}^n\log'(x-a_i)=\sum_{i=1}^n\frac{1}{x-a_i}$. So, for every j, $0=f'(b_j)/f(b_j)=\sum_{i=1}^n\frac{1}{b_j-a_i}$, and thus $\sum_{i=1}^n\sum_{j=1}^{n-1}\frac{1}{a_i-b_j}=\sum_{j=1}^{n-1}\sum_{i=1}^n\frac{1}{a_i-b_j}=\sum_{j=1}^{n-1}0=0$.

5. *Suppose 2024 points are given in the plane with the property that every triangle formed from any three of those 2024 points has area ≤ 1. Prove that all of these points lie in a triangle of area ≤ 4.*

Solution. Choose three points from the given 2024 points that form a triangle of maximal area. (If there are several such triangles, choose one.) Through each of these three points draw the line parallel to the opposite side of the triangle. Any point on the other side of such a line forms, with two opposite vertices, a triangle of larger area, thus all other 2021 points out of 2024 are located inside the large triangle formed by those lines, which triangle has area ≤ 4.



6. *Let T be a triangle in $\mathbb{R}^3$ and let $T_{x,y}$, $T_{y,z}$, $T_{x,z}$ be the projections of T onto the three coordinate planes of $\mathbb{R}^3$. Prove that $\text{area}(T)^2=\text{area}(T_{x,y})^2+\text{area}(T_{y,z})^2+\text{area}(T_{x,z})^2$.*

Solution. Let $T=\triangle ABC$, let $\overrightarrow{AB}=(a_1,a_2,a_3)$ and $\overrightarrow{AC}=(b_1,b_2,b_3)$. Then the area of T is 1/2 of the length of the cross product $\overrightarrow{AB}\times\overrightarrow{AC}$, whose coordinates $a_2b_3-a_3b_2$, $a_3b_1-a_1b_3$ and $a_1b_2-a_2b_1$ are just 2 times the areas of $T_{y,z}$, $T_{x,z}$ and $T_{x,y}$ respectively.