What Is Uniform Convexity?

Abstract

In the L^1 and L^∞ norms, the unit balls are not “round” in the way that the other L^p norms are. As a consequence of this, the L^1 and L^∞ spaces do not have many of the nice properties that the other L^p spaces have; for example, they are not reflexive. In this talk, we will make these feelings precise by defining *uniform convexity* and discussing how it allows us to prove useful properties of functional spaces.

Figure 1 shows the unit balls in $\mathbb{R}^2$ under the L^p norms, for $1 \leq p \leq \infty$.

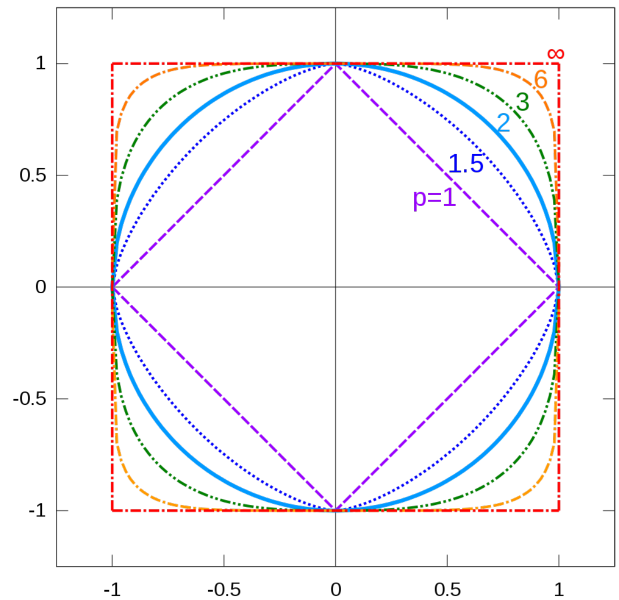

Figure 1: Image due to Quartl on Wikipedia page “Lp_space”

The L^1 and L^∞ norms are not “round” in the sense that their unit balls can be drawn with straight lines, but we can also phrase this by saying that there are norm 1 vectors $x \neq y$ for which the triangle inequality is an equality. I.e. if $x = (1, 0)$ and $y = (0, 1)$, then

$$2 = \|x + y\|_1 = \|x\|_1 + \|y\|_1$$

and if $x = (1, 1)$ and $y = (1, -1)$ then

$$2 = \|x + y\|_\infty = \|x\|_\infty + \|y\|_\infty .$$

Norms for which the triangle inequality is always a strict inequality (when x, y are not colinear) are called strictly convex.

Definition 1

A norm $\|\cdot\|$ is *strictly convex* if for all $x \neq y$ with $\|x\|, \|y\| \leq 1$, we have

$$\left\|\frac{x+y}{2}\right\| < 1.$$

(Equivalently, each point on the boundary of the $\|\cdot\|$-unit ball is an extreme point.)

Strictly convex norms usually make our lives much easier, and make proofs more convenient.

Proposition 2

Let K be a (nonempty) convex, closed and bounded set in a normed vector space. Suppose that this norm is strictly convex. Then K has an *extreme point*, i.e. a point which is not the average of two other points in K.

Proof:

WLOG K is contained in the closed unit ball and there is an $x \in K$ with $\|x\| = 1$. Suppose that there are $y, z \in K$ such that $\frac{y+z}{2} = x$. By the triangle inequality, it must be that $\|y\| = \|z\| = 1$, but then by strict convexity, both y and z must be equal to x. Hence x is an extreme point of K.

Q.E.D.

For the remainder of the talk, we will use X to denote an arbitrary Banach space.

Proposition 3

If X is separable, we can always find an equivalent norm for X which is strictly convex.

The idea here is to start with some norm on X and add to it a norm which is small enough to not change the topology, but will make it strictly convex. For example, if $X = \ell^1(\mathbb{N})$ then define a new norm by

$$|x| := \|x\|_1 + \|x\|_2 .$$

Proposition 3 means that we cannot use strict convexity to tell us anything about the topology of the space.

This suggests that we need a stronger notion.

Definition 4

A norm $\|\cdot\|$ is *uniformly convex* if for each $\epsilon > 0$ there is a $\delta > 0$ such that if x, y have $\|x\|, \|y\| \leq 1$ and

$$\left\|\frac{x+y}{2}\right\| > 1-\epsilon \text{ then } \|x-y\| < \delta.$$

Equivalently, if x_n, y_n are sequences in the (closed) unit ball of $\|\cdot\|$ with $\|x_n + y_n\| \to 2$, then $\|x_n - y_n\| \to 0$.

Exercise (Easy)

A uniformly convex norm is strictly convex. Also, in a finite dimensional space, these two definitions are the same.

Exercise (Also Easy)

Any Hilbert space norm is uniformly convex.

Proposition 5

If X has a uniformly convex norm then any (nonempty) closed convex set will contain an element of minimal norm.

Proof:

Let K be a closed, convex set and WLOG assume $\inf_{x \in K} \|x\| = 1$. Let x_n be a sequence in K with $\|x_n\| \to 1$. For each n, m, $\left\|\frac{x_n + x_m}{2}\right\| \geq 1$ since $\frac{x_n + x_m}{2} \in K$. But by the triangle inequality, $\limsup_{n,m \to \infty} \left\|\frac{x_n + x_m}{2}\right\| \leq 1$. So $\left\|\frac{x_n + x_m}{2}\right\| \to 1$ as $n, m \to \infty$.

Hence $\|x_n - x_m\| \to 0$ by uniform convexity. So (x_n) is Cauchy and so it converges in K.

Q.E.D.

The ideology here is that for convex optimization purposes, strict convexity gives uniqueness, and uniform convexity gives existence (and also uniqueness).

Exercise (Bonus)

The L^p norm is uniformly convex for $1 < p < \infty$, with $\delta = (1 - (\epsilon/2)^r)^{\frac{1}{r}}$, where $r = p$ if $p \geq 2$ and $r = \frac{p}{p-1}$ if $1 < p < 2$.

Hint: Google Clarkson's inequalities.

Now let's take a look at some examples of how arguments using uniform convexity usually go.

Lemma 6

Let X be a uniformly convex space and let $A \in B(X)$ have operator norm 1. If $\left\|\frac{\mathrm{Id}_X + A}{2}\right\|_{B(X)}$ is also equal to 1, then for each $\epsilon > 0$ there is a $v \in X$ with $\|v\|_X = 1$ such that $\|Av - v\| < \epsilon$.

Corollary 7

Let X be a uniformly convex space and let $A_0, A_1, \cdots : X \to X$ each have operator norm 1, with $A_0 = \mathrm{Id}_X$. If for each N,

$$\left\|\frac{1}{N}\sum_{n=0}^{N} A_n\right\|_{B(X)} = 1$$

then there is a sequence of vectors (v_n) such that $\sup_{n \le N} \|A_n v_N - v_N\| \to 0$ as $N \to \infty$.

In particular, we can take $X = \ell^2(\mathbb{Z})$ and $A_n = L^n$, where L is the left shift, so that $\left\|\frac{1}{N}\sum_{n=0}^{N} A_n\right\|_{B(X)} = 1$ by Kesten's criterion for amenability.

Theorem 8 (Milman–Pettis)

If X is uniformly convex, then X is reflexive, i.e. $X^{**} = X$.

Proof:

Let $\xi \in X^{**}$. WLOG $\|\xi\|_{X^{**}} = 1$. We need to show two facts:

- $X \hookrightarrow X^{**}$ and moreover, the closed unit ball of X is weak* dense in the closed unit ball of X^{**}
- For each $\epsilon > 0$, there is an $x \in X \hookrightarrow X^{**}$ such that $\|x - \xi\|_{X^{**}} \le \epsilon$.

The first fact is a statement of Goldstine's Theorem. For the second fact pick any $\epsilon > 0$, and pick δ from the uniform convexity of X. Since $\|\xi\|_{X^{**}} = 1$, we can find $f \in X^*$ with $\|f\|_{X^*} = 1$ such that $|\xi(f)| > 1 - \frac{\delta}{2}$. Now consider the weak* open ball

$$U = \{\eta \in X^{**} : |(\eta - \xi)(f)| < \frac{\delta}{2}\}.$$

By the first fact above, there is an $x \in X \cap U$. To show that this x satisfies $\|x - \xi\|_{X^{**}} \le \epsilon$, we will suppose for the sake of contradiction that it does not.

Then $X \cap U$ is not contained in a closed ball of radius ϵ (in $\|\cdot\|_{X^{**}}$) around x, so there is a point $y \in X \cap U$ with $\|x-y\|_{X^{**}} = \|x-y\|_X \geq \epsilon$. By uniform convexity, this means that $\left\|\frac{x-y}{2}\right\|_X \leq 1-\delta$. But U is convex and so $\frac{x+y}{2} \in U$. This means that

$$|(\frac{x+y}{2} - \xi)(f)| < \frac{\delta}{2}$$

and

$$|\xi(f)| > 1 - \frac{\delta}{2}$$

so

$$(\frac{x+y}{2})(f) > 1-\delta$$

But this is impossible since $\left\|\frac{x-y}{2}\right\|_X \leq 1-\delta$ and $\|f\|_{X^*} = 1$. This is the desired contradiction and the statement of the theorem follows from the second fact and the fact that X is norm closed in X^{**}.

Q.E.D.

Corollary 9

If X is uniformly convex, then every closed, bounded, convex set in X is weakly compact.

Theorem 10 (Kadets–Klee)

Suppose that X is uniformly convex. If $x_n \to x$ weakly in X and $\|x_n\| \to \|x\|$ then $\|x - x_n\| \to 0$.

Proof:

First recall the following fact: If $x_n \to x$ weakly, then $\|x\| \leq \liminf_n \|x_n\|$. This is true since for each $f \in X^*$ we have $f(x_n) \to f(x)$ and so $|f(x)| = \liminf_n |f(x_n)| \leq \liminf_n \|f\| \cdot \|x_n\|$. But the fact follows since $\|x\| = \sup_{\substack{f \in X^* \\ \|f\|_{X^*}=1}} |f(x)|$.

WLOG each x_n has norm 1 (divide by $\|x_n\|$ if necessary).

Now to show the statement of the theorem. We know that $\frac{x_n+x}{2} \to x$ weakly, and so

$$1 = \|x\| \leq \liminf_n \left\|\frac{x_n+x}{2}\right\|.$$

But each $\frac{x_n+x}{2}$ has norm at most 1, so $\left\|\frac{x_n+x}{2}\right\| \to 1$. By uniform convexity, this means that $\|x_n - x\| \to 0$.

Q.E.D.

Theorem 11 (Browder's Fixed Point Theorem)

Let X be a uniformly convex space and let K be a (nonempty) convex, closed, bounded subset of X. Let $U : K \to K$ be a *weak contraction*, meaning $\|U(x) - U(y)\| \leq \|x - y\|$ for all $x, y \in K$. Then U has a fixed point.

Proof:

Let $\mathscr{F}$ be the collection of nonempty, closed, convex subsets of K which are U invariant, ordered by inclusion. $K \in \mathscr{F}$ and if (C_λ) is a descending chain in $\mathscr{F}$ then we may use Cantor's theorem to say that $\bigcap_\lambda C_\lambda$ is nonempty since each C_λ is weakly compact. By Zorn's lemma, $\mathscr{F}$ has a minimal element, call it C. Now let's show that C is a singleton, since this will complete the proof of the theorem.

First note that by minimality, the closure of the convex hull of $U(C)$ is C, i.e. $\overline{\text{Conv}(U(C))} = C$.

Suppose that C has nonzero diameter. WLOG C has diameter 1.

Claim: We can find an $x \in C$ and $r < 1$ such that each point $y \in C$ has $\|x - y\| \leq r$.

To see that this is true pick $0 < \epsilon < \frac{1}{2}$, pick δ from uniform continuity and pick $x_1, x_2 \in C$ such that $\|x_1 - x_2\| > \delta$. Let $x = \frac{x_1 + x_2}{2}$. Then for any $y \in C$,

$$\|x - y\| = \left\| \frac{(x_1 - y) + (x_2 - y)}{2} \right\|$$

is smaller than $1 - \epsilon$ since

$$\|(x_1 - y) - (x_2 - y)\| = \|x_1 - x_2\| > \delta.$$

Taking $r = 1 - \epsilon$ proves the claim.

Now let

$$C' = \bigcap_{y \in C} \{w \in C : \|w - y\| \leq r\}.$$

C' is convex, closed and nonempty since $x \in C'$.

C' is also U invariant since if $u \in C'$ then for any $y \in C$ we can show that $\|U(u) - y\| \leq r$.

To this end, pick $y \in C$ and recall that the convex hull of $U(C)$ is dense in C, i.e. $\text{dist}(\text{Conv}(U(C)), y) = 0$. Also recall that U is a weak contraction, which gives that

$$\begin{aligned}
\|U(u) - y\| &\leq \text{dist}(U(u), \text{Conv}(U(C))) + \text{dist}(\text{Conv}(U(C)), y) \\
&= \text{dist}(U(u), \text{Conv}(U(C))) + 0 \\
&\leq \text{dist}(U(u), U(C)) \leq \text{dist}(u, C) \leq r
\end{aligned}$$

which means that $U(u) \in C'$. But C' is properly contained in C since the diameter of C' is r which is less than the diameter of C. This is the desired contradiction, so C must be a singleton.

Q.E.D.

Theorem 12 (Browder)

Let X be a uniformly convex space and let K be a (nonempty) convex, closed, bounded subset of X. Let $(U_i)_{i\in I}$ be an arbitrary family of commuting weak contractions from K to K. Then there is a point in K which is fixed by each U_i.

Proof:

Claim: For each i, the fixed point set of U_i is convex.

To see why this is true pick $a, b \in \text{Fix}(U_i)$ and let c be a point on the line between them. Then

$$\|a - U_i(c)\| = \|U_i(a) - U_i(c)\| \leq \|a - c\|$$

and

$$\|b - U_i(c)\| = \|U_i(b) - U_i(c)\| \leq \|b - c\|$$

so $U_i(c)$ is at least as close to both a and b as c is. Additionally,

$$\begin{aligned}\|a-b\| \leq \|a - U_i(c)\| + \|U_i(c) - b\| &= \|U_i(a) - U_i(c)\| + \|U_i(c) - U_i(b)\| \\ &\leq \|a-c\| + \|c-b\| = \|a-b\|\end{aligned}$$

hence $U_i(c)$ is also on the line between a and b. Hence $U_i(c)$ must be c. This proves the claim.

Now the collection $(\text{Fix}(U_i))_{i\in I}$ is a family of closed, convex, nonempty subsets of X. That makes each of these sets weakly compact (since K is weakly compact), and so if we can show that this collection has the finite intersection property, then we are done.

Claim: Let $U_1, \ldots, U_n$ be finitely many elements from this family. Then $\bigcap_{i=1}^{n} \text{Fix}(U_i)$ is nonempty.

To see that this is true, note that if $u \in \text{Fix}(U_i)$ then for any j we have

$$U_i U_j(u) = U_j U_i(u) = U_j(u)$$

meaning that $U_j(u) \in \text{Fix}(U_i)$. So each $\text{Fix}(U_i)$ is invariant under each U_j.

Now we will proceeded by induction on n. Theorem 11 shows the case $n = 1$. Assuming that $\bigcap_{i=1}^{n-1} \text{Fix}(U_i)$ is nonempty, we can note that this set is U_n invariant and applying Theorem 11 gives that $\bigcap_{i=1}^{n} \text{Fix}(U_i)$ is nonempty.

Q.E.D.

Theorem 13 (Edelstein)

Assume X is only strictly convex and that K is compact and convex. If $U : K \to K$ is a weak contraction then we can find a fixed point of U by picking any $x_0 \in K$ and taking the limit of the sequence (x_n) where $x_{n+1} = \dfrac{x_n + Ux_n}{2}$.

Proof:

First note that if u is any fixed point for U (which exists by Schauder's fixed point theorem) and if v is any non-fixed point then we have

$$\left\|\frac{v+U(v)}{2}-u\right\|=\left\|\frac{v+Uv}{2}-\frac{u+Uu}{2}\right\|<\frac{\|v-u\|+\|Uv-Uu\|}{2}\leq\|v-u\|\,. \tag{1}$$

(this is where we use the assumption of strict convexity)

Since K is compact, (x_n) has a limit point $p \in K$. We can suppose that $U(x_n) \neq x_n$ for all n, since otherwise we would be done.

If we can take $u = p$, then this means that x_n converges to p by equation (1).

Suppose for the sake of contradiction that $U(p) \neq p$. Let $q = \dfrac{p+U(p)}{2}$, let $u \in K$ be any fixed point and let $r = \dfrac{\|p-u\|-\|q-u\|}{2}$, which is positive by equation (1). Let B be the ball of radius r around q. Since p is a limit point of (x_n) there will be arbitrarily large values of n for which x_{n+1} is within r of q. But then

$$\|x_{n+1}-y\|\leq\|x_{n+1}-q\|+\|q-y\|<r+\|q-y\|=\frac{\|p-u\|+\|q-u\|}{2}$$

and also

$$\|x_{n+1}-p\|\geq\|y-p\|-\|x_{n+1}-y\|>\|y-p\|-\frac{\|p-u\|+\|q-u\|}{2}=r$$

This holds for arbitrarily large values of n and so this contradicts the fact that p is a limit point of (x_n). So we are done.

Q.E.D.

References

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