Notes What is... Schubert Calculus?

Introduction

Consider the following questions:

1. (Easy) Given two points in $\mathbb{R}^2$, how many lines pass through them?

2. (Easy) Given two lines in $\mathbb{R}^2$, how many points are contained in both lines?

   * Note, the answer to #2 depends on the position of the lines.

3. (Medium) Given four lines in $\mathbb{R}^3$, how many lines are there which intersect all 4?

   * Note, the answer here also depends on position, we have to invoke that the original lines are in "general enough" position. What does this mean?

4. (Hard) How many k-dim'l subspaces of $\mathbb{C}^n$ intersect each of a given set of $k\cdot(n-k)$ fixed subspaces of dimension $n-k$ non-trivially?

In 1879 Hermann Schubert wrote a book "Kalkül der abzählend Geometrie" (roughly Calculus of Enumerative Geometry) developing a calculus for solving these sorts of problems.

In this original treatment of the subject, Schubert would have played around with "general enough" position and invoke what he called the "conservation of number" to argue that the answer would remain constant for sufficiently non-degenerate cases. For example, let's look at question #3 through this lens:

Let l_1, l_2, l_3, and l_4 be distinct lines in $\mathbb{R}^3$. Position them such that $l_1 \cap l_2 = P$, $l_3 \cap l_4 = Q$, and the planes formed by (l_1 and l_2) $= \tilde{P}_{pe}$ and by (l_3 and l_4) $\tilde{Q}_{pe}$ are not parallel. Then the lines $\overleftrightarrow{PQ}$ and $(\tilde{Q}_{pe} \cap \tilde{P}_{pe})$ are the two solutions to the question in this configuration. Schubert would say this implies that this is the answer, and indeed it is.

Through this method, Schubert put forth many correct answers to these types of questions, but many found this non-rigorous.

David Hilbert was one person who found this non-rigorous (it may have led to incorrect solutions?), so in his famous list of 23 problems in 1900, putting Schubert's enumerative calculus on a rigorous foundation was problem #15. This has been largely answered since then in the subject of what is now known as Schubert Calculus.

Overview

Just in case we do not make it all the way through to working a complete example, I will now outline the very rough method for doing examples of such questions.

- First, we introduce 2 important spaces: Projective space and the Grassmannian. These spaces have very nice topological structures; they are varieties, manifolds, and cell complexes. These spaces more or less are comprised of linear subspaces.

- Second, we consider certain subsets of these spaces which correspond to our conditions (whatever the question asks) and...

- Third, find the "intersection" of these subsets (using the cohomology) to give our answer.

If we understand the set-up, the overview, and these important spaces, I will consider this a successful talk. Everything else will be icing on the cake.

Projective Space

This is probably a review for many of us:

<u>Def^{n}</u>: Let k be a field, then the n-dim'l projective space over k, denoted $\mathbb{P}^n_k$ is as a set

$$\frac{k^{n+1} \setminus \{\vec{0}\}}{\sim}$$

where $(x_0, \dots, x_n) \sim (y_0, \dots, y_n)$ iff $\exists\, a \in k$ s.t. $x_i = a y_i \;\forall\, i \in \{0, \dots, n\}$.

If k^{n+1} has a topology (such as the standard topology for $k = \mathbb{R}$ or $\mathbb{C}$, then we can give $\mathbb{P}^n_k$ the quotient topology.

<u>Remark</u>: This can also be thought of as the set of lines through the origin in k^{n+1}. Indeed

$$[(x_0, \dots, x_n)] \longleftrightarrow \{\text{line going through } 0 \text{ and } (x_0, \dots, x_n) \text{ in } k^{n+1}\}$$

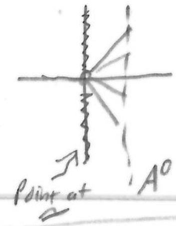
<u>Remark</u>: We will denote $[(x_0, \dots, x_n)]$ by $(x_0 : \dots : x_n)$.

<u>Def^{n}</u>: The i^{th} affine patch of $\mathbb{P}^n$ $(i \leq n)$ is the "subset" where the i^{th} coordinate is 1, denote $A^i(\mathbb{P}^n)$. These form a cover

<u>Ex</u>: $\mathbb{P}^1_{\mathbb{R}}$, $\{(x_0 : x_1)\}$, $A^0(\mathbb{P}^1_{\mathbb{R}}) = \{(1 : x_1)\}$

and we can see that $\underset{\;}{\mathbb{P}^1_{\mathbb{R}}} = \underset{\underset{\mathbb{I}}{\cong}}{A^0(\mathbb{P}^1_{\mathbb{R}})} \sqcup \underset{\underset{\{pt\}}{\cong}}{(0:1)}$

⑤



which shows both that $\mathbb{P}^1_{\mathbb{R}}$ topologically is S^1, and alludes to a cell-complex structure.

<u>Lastly:</u> A line in $\mathbb{P}^2(\mathbb{R})$ is the solution to $ax+by+cz=0$.

So if we take $2x+3y+z=0$ "parallel lines"
$2x+3y-z=0$

They intersect at $(3:-2:0)$ in $\mathbb{P}^2$.

☆ So distinct lines in projective space intersect.

[So earlier question 2 has no ambiguity.]

⌈We will translate the questions back and forth between projective space and not projective space because of this.

For example:

{How many points are contained in two distinct lines in $\mathbb{P}^2_{\mathbb{C}}$} $\longleftrightarrow$ {How many 1 dim subspaces are contained in two distinct two dim'l subspaces in $\mathbb{C}^3$}

{How many lines through 4 given lines in $\mathbb{P}^3$} $\longleftrightarrow$ {How many 2-dim'l subspaces (V) in $\mathbb{C}^4$ intersect 4 given 2-dim'l subspaces $V_1, \ldots, V_4$ in $\mathbb{C}^4$ (s.t. $\dim(V \cap V_i) \geq 1$)}

Grassmannians

Def$^{\underline{n}}$: The Grassmannian is the set of all k-dim'l subspaces of $\mathbb{C}^n$

Topologically, $Gr(n,k) = \underline{\{\text{full rank } k \times n \text{ matrices}\}}_{\sim}$

where $\sim$ is equivalence by elementary row operations (scaling, adding, swapping).

Ex: $Gr(n,1) = \mathbb{P}^n_{\mathbb{C}}$ the only row operation on $1 \times n$ matrices is scaling.

Ex: In $Gr(7,3)$,

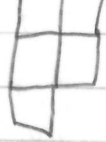
$$\begin{bmatrix} 0 & -1 & -3 & -1 & 6 & -4 & 5 \\ 0 & 1 & 3 & 2 & -7 & 6 & -5 \\ 0 & 0 & 0 & 2 & -2 & 4 & -2 \end{bmatrix} \in \overbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 1 & * & 0 & * & * & 0 \end{bmatrix}}^{\text{Schubert cell}}$$

Note: Notation is different in many sources.

Note: The Plücker embedding shows $Gr(n,k) \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}$.

Def$^{\underline{n}}$: A partition, λ, is $(\lambda_1, \dots, \lambda_r)$ $\lambda_1 \geq \dots \geq \lambda_r$, $\ell(\lambda) = r$, $|\lambda| = \sum \lambda_i$.

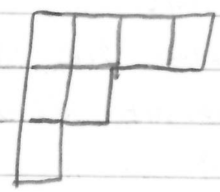
Def$^{\underline{n}}$: Given a partition, λ, the Young diagram associated to λ is best explained via example

Ex. $(2,2,1) \longleftrightarrow$  **Note**: Different notation.

"Important subsets"

There is a correspondence between $\boxed{\text{Schubert Cells}}$ and "partitions" obtained by

1) Blocking out $k \times k$ staircase
2) reading the partition (to the pivots)

Ex: $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 1 & * & 0 & * & * & 0 \end{bmatrix} \longleftrightarrow$  $(4,2,1)$

☆ **Note**: The only admissible partitions in this correspondence must fit inside the $k \times (n-k)$ rectangle (I will call it the ambient rectangle).

i.e. $\ell(\lambda) \le k \quad , \quad \lambda_1 \le (n-k)$.

Further, you can see that the boxes and stars fill up this ambient rectangle.

Finally, rigorously, we can define Schubert cells and Schubert Varieties.

Def$^{\underline{n}}$: Given $\lambda \subset$ Ambient, the Schubert cell Ω^o_λ is defined as follows (given $e_1, \dots, e_n$ a basis $[e_1 = (0, \dots, 0, 1)]$)

$$\Omega^o_\lambda = \{ V \in Gr(n,k) \mid \dim(V \cap \langle e_1, \dots, e_r \rangle) = i \text{ , for } n-k+i-\lambda_i \le r \le n-k+i-\lambda_{i+1} \}$$

Further, we can see that $\dim(\Omega_\lambda^\circ) = k(n-k) - |\lambda|$

<u>Def$^{\underline{n}}$</u>: The Schubert Variety $\Omega_\lambda = \overline{\Omega_\lambda^\circ} \subset Gr(n,k)$ is

$$\Omega_\lambda = \{ V \in Gr(n,k) \mid \dim(V \cap \langle e_1, \dots, e_{n-k+i-\lambda_i} \rangle) \geq i \}$$

Note, I will not go into this much, but the Schubert cells does actually yield a CW-complex structure and Schubert varieties are actual varieties. Further, one can decompose Schubert varieties into Schubert cells

<u>Ex</u>: Consider $\mathbb{P}^5 = Gr(6,1)$. Then

$$\Omega_{\square\square} = \{ V \in Gr(6,1) \mid \dim(V \cap \langle e_1, \dots, e_4 \rangle) \geq 1 \}$$

So, $V \subset \langle e_1, \dots, e_4 \rangle$ (recall $e_1 = (0, \dots, 0, 1)$, and so on)

So, each point in $\Omega_{\square\square}$ can be written in one of the following forms:

$$(0:0:1:*:*:*) = \Omega^\circ_{\square\square}$$
$$(0:0:0:1:*:*) = \Omega^\circ_{\square\square\square}$$
$$(0:0:0:0:1:*) = \Omega^\circ_{\square\square\square\square}$$
$$(0:0:0:0:0:1) = \Omega^\circ_{\square\square\square\square\square}$$

You can see that $\Omega^\circ_{\square\square}$ picks up a lot when the closure is taken.

Now, circling back to our questions, consider

$\Omega_{\square} \subset Gr(4,2)$.

$$\Omega_{\square} = \{ V \in Gr(4,2) \mid \dim(V \cap \langle e_1, e_2 \rangle) \geq 1 \}$$

All 2 dimensional subspaces which intersect a fixed 2-D subspace in dimension at least 1.

This is almost what we want to answer question 3 (by working in one dimension higher in $Gr(4,2)$), but we need 4 different fixed subspaces. (this equivalence is via the quotient map $\mathbb{C}^4 \twoheadrightarrow \mathbb{P}^3$).

How do we make this happen? Flags.

Def$^{\underline{n}}$: A complete flag $F_\bullet$ in $\mathbb{C}^n$ is

$$0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n = \mathbb{C}^n$$
where $\dim(F_i) = i$.

A Schubert Variety with respect to a flag is

$$\Omega_\lambda(F_\bullet) = \{ V \in Gr(n,k) \mid \dim(V \cap F_{n-k+i-\lambda_i}) \geq i \}$$

$$\Omega^\circ_\lambda(F_\bullet) = \{ V \in Gr(n,k) \mid \dim(V \cap F_r) = i \text{ for (same conditions)} \}$$

Now, we have reduced question 3 to showing that for 4 distinct flags (indeed we need $F_2^{(i)} = V_i$) that

$$\left|\Omega_{\square}(F_\bullet^1) \cap \Omega_{\square}(F_\bullet^2) \cap \Omega_{\square}(F_\bullet^3) \cap \Omega_{\square}(F_\bullet^4)\right| = 2.$$

There is a way to do this.

Sprint To the Finish Line.

Def$^{\underline{n}}$: • Two subspaces meet transversely ($V, W \subseteq \mathbb{C}^n$) if $\dim(V \cap W) = \max(0, \dim(V) + \dim(W) - n)$

• Two flags are transverse if $F_i \cap E_{n-i} = \{0\}$ $\forall i$.

Def$^{\underline{n}}$: Two partitions λ, μ are complementary in the ambient rectangle if (after rotating μ) they fill the ambient rectangle.

Ex: $\lambda = (4,2,1)$, $\mu = (3,2)$ in $Gr(7,3)$

0	0	0	0
0	0	*	*
0	*	*	*

we say $\mu^c = \lambda$

Thm: (Duality) Let $F_\bullet$ and $E_\bullet$ be transverse flags in $\mathbb{C}^n$, let λ, μ be partitions with $|\lambda| + |\mu| = k(n-k)$. Then in $Gr(n,k)$

$$|\Omega_\lambda(F_\bullet) \cap \Omega_\mu(E_\bullet)| = \begin{cases} 1 & \text{if } \lambda = \mu^c \\ 0 & \text{else} \end{cases}$$

Cor: Given transverse flags in $Gr(n+1,2)$, we can see that $|\Omega_{(n-1)}(F_\bullet) \cap \Omega_{(n-1)}(E_\bullet)| = 1$ by the duality thm. Unravelling this, we can see that this means that given any two points in $\mathbb{P}^n$, the intersect at exactly one point.

Facts leading to more general results:

- Given $\Omega_\lambda(F_\bullet) \subseteq Gr(n,k)$, we can obtain

  $$\sigma_\lambda := [\Omega_\lambda(F_\bullet)] \in H^{2|\lambda|}(Gr(n,k)).$$

  This provides a $\mathbb{Z}$-basis for $H^*(Gr(n,k))$ (for λ fitting inside the ambient rectangle), and further,

  $$\sigma_\lambda \cdot \sigma_\mu = [\Omega_\lambda(F_\bullet) \cap \Omega(E_\bullet)]$$

  for transverse flags.

- Thm: $\boxed{\text{If } B \text{ is the ambient rectangle, } \sigma_B \text{ generates the top cohomology, so if we have admissible partitions } \lambda^{(1)}, \dots, \lambda^{(r)} \text{ with } \sum_i |\lambda^{(i)}| = k(n-k), \text{ then } \sigma_{\lambda^1} \cdots \sigma_{\lambda^r} = c^B_{\lambda^1, \dots, \lambda^r}\, \sigma_B \text{ and } |\Omega_{\lambda_1} \cap \cdots \cap \Omega_{\lambda_r}| = c^B_{\lambda^1, \dots, \lambda^r}.}$

- <u>Def</u>

Def$^{\underline{n}}$ Given λ, the schur function $s_\lambda = \sum_T x^T$ for semi-standard young tableaux T

  <u>Ex:</u> $$s_{\tiny\begin{smallmatrix}\square&\square\\\square\end{smallmatrix}} = x^{\begin{smallmatrix}1&1\\2\end{smallmatrix}} + x^{\begin{smallmatrix}1&2\\2\end{smallmatrix}} + x^{\begin{smallmatrix}1&2\\3\end{smallmatrix}} + x_1^2x_2 + x_1x_2^2 + x_1x_2x_3 + \cdots$$

- Thm: $\boxed{\begin{array}{ccc} H^*(Gr(n,k)) & \longrightarrow & \Lambda(x_1, x_2, \dots) / (s_\lambda \mid \lambda \not\subseteq B) \\ [\Omega_\lambda] & \longmapsto & s_\lambda \end{array} \text{ is an isomorphism}}$

Thus, we can compute this with the Littlewood-Richardson rule:

Let B be the $k \times (n-k)$ rect., let $\lambda^{(1)}, \dots, \lambda^{(m)}$ be partitions fitting inside B s.t. $|B| = \sum_i |\lambda^{(i)}|$.

For generic flags, $c^B_{\lambda^1 \dots \lambda^m} = |\Omega_{\lambda^1} \cap \dots \cap \Omega_{\lambda^m}|$

= to the total number of chains of LR tableaux of contents $\lambda^1, \dots, \lambda^m$ with total shape B.

An LR Tableaux is a SSYT whose reading word (concatenate the contents from bottom to top) is Yamanouchi (each suffix has $|i| \geq |j|$ for $i < j$).

Ex: $\lambda^1 = (2,1)$, $\lambda^2 = (2,1)$, $\lambda^3 = (3,1)$, $\lambda^4 = (2)$

1	1	1	1
2	2	1	1
1	2	1	1

There are 5

There are 5

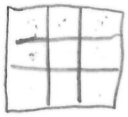
Ex: $\sigma_\square \cdot \sigma_\square \cdot \sigma_\square \cdot \sigma_\square = 2\,\sigma_{\boxplus}$

i.e. $c^{\boxplus}_{\square,\square,\square,\square}$

1_1	1_2
1_3	1_4

,

1_1	1_3
1_2	1_4



$k=2 \quad n=4$

$(2(2)-2)!$

$(3\cdot 3-3)! = 6!$

$$\frac{9!}{5\cdot 4\cdot 3\cdot 2\cdot 1 \cdot 4\cdot 3\cdot 3\cdot 2}$$

$\cancel{9}\cdot\cancel{8}\cdot 7\cdot 6$

$7\cdot 6$

The "hard" problem is $=$ # SYT of shape B

$$= \left(k(n-\cancel{k})-2\right)!$$