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1 Isoperimetric Inequality in $\mathbb{R}^2$

1.1 Prerequisite Information

Definition 1.1.1: let $S \subset \mathbb{R}^2$ be a bounded region whose boundary is a C^2 simple closed curve $c(t) : \mathbb{R} \to \mathbb{R}^2$ with period ℓ. Then we can define the perimiter of S to be,

$$\mathrm{P}(S) = \int_0^{\ell} ||c'(t)||dt$$

Likewise if $c(t) = (x(t), y(t))$ we can define the area of S to be,

$$\mathrm{A}(S) = \frac{1}{2} \int_0^{\ell} x(t)y'(t) - x'(t)y(t)dt$$

Remark 1.1.2: Note that our formula for area is a direct corrolary of Greens theorem because

$$\frac{1}{2} \int_0^{\ell} x(t)y'(t) - x'(t)y(t)dt = \frac{1}{2} \int_{c(t)} xdy - ydx = \iint_S 1dA = \mathrm{A}(S)$$

Theorem 1.1.3: For all C^2 periodic functions $f(x) : \mathbb{R} \to \mathbb{C}$ with period 2π there exists a sequence c_k such that,

$$\mathscr{F}_N(x) = \sum_{k=-N}^{N} c_k e^{ikx}$$

Converges uniformly to $f(x)$ as $N \to \infty$. So we will notate this as,

$$\mathscr{F}(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} = f(x)$$

Remark 1.1.4: Let $\Re(z)$ is the Real part of z and $\Im(z)$ is the imaginary part of z. The complex plane $\mathbb{C}$ can be identified with $\mathbb{R}^2$ by identifying $(x, y) \in \mathbb{R}^2$ with z by $(x, y) = (\Re(z), \Im(z))$.

1.2 Proof of The Isoperimetric Inequality in $\mathbb{R}^2$

Theorem 1.2.1: Let $S \subset \mathbb{R}^2$ be a bounded set which has a boundary that is a C^2 simple closed curve then,

$$4\pi \mathrm{A}(S) \leq \mathrm{P}(S)^2$$

Where equality holds if and only if S is a circle.

Proof. Firstly let $c(t) = (x(t), y(t))$ parametrize the boundary of S by arc length and let $\mathrm{P}(t) = \ell$ be the arc length of $c(t)$. This means that $c(t)$ is ℓ periodic, so define $z(t) = x(\frac{\ell t}{2\pi}) + iy(\frac{\ell t}{2\pi})$ which means $z(t)$ is 2π periodic. Now we will express ℓ^2 in terms of the fourier coefficents of z. We can see,

$$\int_0^{2\pi} ||z'(t)||^2 dt = \int_0^{2\pi} \frac{\ell^2}{4\pi^2} ||c'(t)||^2 dt = \frac{\ell^2}{2\pi}$$

Now in terms of the fourier coefficents we have,

$$\int_0^{2\pi} ||z'(t)||^2 dt = \int_0^{2\pi} z'(t) \cdot \overline{z'(t)} dt = \int_0^{2\pi} \sum_{k,j=-\infty}^{\infty} kjc_k\overline{c_j}e^{i(k-j)t} = \sum_{k,j=-\infty}^{\infty} \int_0^{2\pi} kjc_k\overline{c_j}e^{i(k-j)t} dt$$

$$= 2\pi \sum_{k=-\infty}^{\infty} k^2 ||c_k||^2$$

Now we can see,

$$\int_0^{2\pi} i \cdot z(t) \cdot \overline{z'(t)} dt = \int_0^{2\pi} \left(ix(\frac{\ell t}{2\pi}) - iy(\frac{\ell t}{2\pi}) \right) \frac{\ell}{2\pi} \left(x'(\frac{\ell t}{2\pi}) - iy(\frac{\ell t}{2\pi}) \right) dt$$

$$= \int_0^{\ell} x(s)y'(s) - x'(s)y(s) ds + i \int_0^{\ell} x(s)x'(s) + y(s)y'(s) ds = 2\mathrm{A}(S)$$

Also notice that,

$$\int_0^{2\pi} i \cdot z(t) \cdot \overline{z'(t)} dt = \int_0^{2\pi} \sum_{k,j=-\infty}^{\infty} c_k\bar{c_j}je^{i(k-j)t} = \sum_{k,j=-\infty}^{\infty} \int_0^{2\pi} c_k\bar{c_j}je^{i(k-j)t} = 2\pi \sum_{k=-\infty}^{\infty} k||c_k||^2$$

Thus we have the chain of Inequalities,

$$2\mathrm{A}(S) = 2\pi \sum_{k=-\infty}^{\infty} k||c_k||^2 \leq 2\pi \sum_{k=-\infty}^{\infty} k^2||c_k||^2 = \frac{\ell^2}{2\pi}$$

Which gives,

$$4\pi \mathrm{A}(S) \leq \mathrm{P}(S)^2$$

Note that we have, $\sum_{k=-\infty}^{\infty} k||c_k||^2 = 2\pi \sum_{k=-\infty}^{\infty} k^2||c_k||^2$ if and only if $||c_k|| = 0$ for all $k \neq 0, 1$. Which means that $4\pi \mathrm{A}(S) = \mathrm{P}(S)^2$ if and only if the boundary of S is parametrized by $c_0 + c_1e^{it}$ which is the equation of a circle in $\mathbb{C}$. $\square$

2 Isoperimetric Inequality on Convex subsets of $\mathbb{R}^n$

2.1 Introduction to Convex Geometry

Definition 2.1.1: We call $S \subset \mathbb{R}^n$ Convex if for all $x, y \in S$ we have,

$$(1-\lambda)x + \lambda y \in S, \qquad 0 \leq \lambda \leq 1$$

Additionally we can see that if $A, B \subset \mathbb{R}^n$ are convex then,

$$\lambda B = \{x \in \mathbb{R}^n : \exists b \in B \text{ where } x = \lambda b\}$$

and,

$$A + B = \{x \in \mathbb{R}^n : \exists a \in A, b \in B \text{ where } x = a + b\}$$

are also convex. The second operation will be refered to as the Minkowski sum of A and B.

Definition 2.1.2: Let $A \subset \mathbb{R}^n$ is a Convex Body if A is non-empty, compact, convex. We will denote the set of all convex bodies in $\mathbb{R}^n$ as $\mathscr{K}^n$ and all of the convex bodies with non-empty interior as $\mathscr{K}_0^n$.

Definition 2.1.3: A function $f : \mathbb{R}^n \to \mathbb{R}$ is called convex if for all $x, y \in \mathbb{R}^n$ and all $0 \leq \lambda \leq 1$ f satisfies,

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)$$

Likewise a function f is concave if for all $x, y \in \mathbb{R}^n$ and all $0 \leq \lambda \leq 1$ f satisfies,

$$f((1-\lambda)x + \lambda y) \geq (1-\lambda)f(x) + \lambda f(y)$$

Notice that if f is convex iff $-f$ is concave.

Definition 2.1.4: Let $S \subset \mathbb{R}^n$ be convex then we can denote the Volume of S as,

$$\text{Vol}(S) = \int \cdots \int_S 1 dV$$

Likewise we can denote the surface area or perimiter of S as,

$$\text{P}(S) = \lim_{h \to 0} \frac{\text{Vol}(S + hB_n) - \text{Vol}(S)}{h}$$

Where $B_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq 1\}$ is the unite ball in $\mathbb{R}^n$

Remark 2.1.4: We can notice that we have the following properties:

- $\text{Vol}(\lambda S) = |\lambda|^n \text{Vol}(S)$ for all $\lambda \in \mathbb{R}$
- $\text{Vol}(A + B) \leq \text{Vol}(A), \text{Vol}(B)$ for $A, B \in \mathscr{K}_0^n$

2.2 Volume and Perimiter of B_n

This section will detour us from the Isoperimetric Inequality in $\mathbb{R}^n$, but demonstrates our definitions of perimiter and volume.

Proposition 2.2.1: If $B_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq 1\}$, then

$$\text{P}(B_n) = n\text{Vol}(B_n)$$

Proof. We have that,

$$\begin{aligned}\text{P}(B_n) &= \lim_{h\to 0} \frac{\text{Vol}(B_n + hB_n) - \text{Vol}(B_n)}{h} \\ &= \lim_{h\to 0} \frac{\text{Vol}((1+h)B_n) - \text{Vol}(B_n)}{h} \\ &= \text{Vol}(B_n) \lim_{h\to 0} \frac{|1+h|^n - 1}{h} \\ &= n\text{Vol}(B_n)\end{aligned}$$

□

Remark 2.2.2: We can see that this formula agrees with the traditional formula for the surface area and Volume of B_3 where $\text{P}(B_3) = 4\pi = 3\text{Vol}(B_3)$ with $\text{Vol}(B_3) = \frac{4\pi}{3}$.

2.3 The Brunn-Minkowski Theorem

The following theorem is an important theorem from convex geometry, which will be crucial in the proof of the Isoperimetric Inequality.

Theorem 2.3.1 Let $K_0, K_1 \in \mathscr{K}^n$ be two convex bodies and $\lambda \in [0,1]$ then,

$$\text{Vol}((1-\lambda)K_0 + \lambda K_1)^{\frac{1}{n}} \geq (1-\lambda)\text{Vol}(K_0)^{\frac{1}{n}} + \lambda\text{Vol}(K_1)^{\frac{1}{n}}$$

Where equality holds for $\lambda \in (0,1)$ if and only if K_0 and K_1 are contained in parallel hyperplanes or $K_1 = sK_0 + t$ for $s \geq 0$ and $t \in \mathbb{R}^n$.

Corrolary 2.3.2 Let $K_0, K_1 \in \mathscr{K}^n$ and $I = [0,1]$ Then the function $f : I \to \mathbb{R}$ given by,

$$f(\lambda) = \text{Vol}((1-\lambda)K_0 + \lambda K_1)^{1/n}$$

Is concave. f is linear if and only if K_0 and K_1 are contained in parallel hyperplanes or $K_1 = sK_0 + t$ for $s \geq 0$ and $t \in \mathbb{R}^n$

Proof.

$$\begin{aligned}f((1-\lambda)x + \lambda y) &= \text{Vol}((1 - (1-\lambda)x - \lambda y)K_0 + ((1-\lambda)x + \lambda y)K_1)^{1/n} \\ &= \text{Vol}((1-\lambda)(1-x)K_0 + (1-\lambda)xK_1 + \lambda(1-y)K_0 + \lambda yK_1)^{1/n} \\ &\geq (1-\lambda)\text{Vol}((1-x)K_0 + xK_1)^{1/n} + \lambda\text{Vol}((1-y)K_0 + yK_1)^{1/n} \\ &= (1-\lambda)f(x) + \lambda f(y)\end{aligned}$$

□

Lemma 2.3.3 Let $f : I \to \mathbb{R}$ be a smooth concave function such that $f'(0) = f(1) - f(0)$ holds, then f is linear.

2.4 Proof of The Isoperimetric Inequality in $\mathbb{R}^n$

Theorem 2.4.1 Let $K \in \mathscr{K}_0^n$. Then,

$$\mathrm{P}(K) \geq n\mathrm{Vol}(B_n)^{\frac{1}{n}}\mathrm{Vol}(K)^{1-\frac{1}{n}}$$

Equality holds iff K is a ball.

Proof. Let $K \in \mathscr{K}_0^n$. We have $\mathrm{Vol}(K) \neq 0$. Let $\epsilon = \frac{t}{1-t}$ then,

$$\begin{aligned}
\mathrm{P}(K) &= \lim_{t\to 0} \frac{\mathrm{Vol}(K + \frac{t}{1-t}B_n) - \mathrm{Vol}(K)}{\frac{t}{1-t}} \\
&= \lim_{t\to 0} \frac{\mathrm{Vol}((1-t)K + tB_n) - (1-t)^n\mathrm{Vol}(K)}{(1-t)^{n-1}t} \\
&= \lim_{t\to 0} \left(\frac{\mathrm{Vol}((1-t)K + tB_n) - \mathrm{Vol}(K)}{t} + \frac{(1-(1-t)^n)\mathrm{Vol}(K)}{t} \right) \\
&= \lim_{t\to 0} \left(\frac{\mathrm{Vol}((1-t)K + tB_n) - \mathrm{Vol}(K)}{t} \right) + n\mathrm{Vol}(K)
\end{aligned}$$

This means,

$$\mathrm{P}(K) - n\mathrm{Vol}(K) = \lim_{t\to 0} \left(\frac{\mathrm{Vol}((1-t)K + tB_n) - \mathrm{Vol}(K)}{t} \right)$$

Now consider the function $f(t) := \mathrm{Vol}((1-t)K + tB_n)^{\frac{1}{n}}$ and see,

$$f'(t) = \frac{1}{n}\mathrm{Vol}((1-t)K + tB_n)^{\frac{1}{n}-1} \cdot \frac{d}{dt}\Big(\mathrm{Vol}((1-t)K + tB_n)\Big)$$

Now from the above calculation we have,

$$f'(0) = \frac{1}{n}\mathrm{Vol}(K)^{\frac{1}{n}-1}(\mathrm{P}(K) - n\mathrm{Vol}(K))$$

Now by corrolary 2.3.2 it follows f is concave on $[0, 1]$, hence $f'(0) \geq f(1) - f(0)$. This gives us that,

$$\frac{1}{n}\mathrm{Vol}(K)^{\frac{1}{n}-1}\mathrm{P}(K) - \mathrm{Vol}(K)^{\frac{1}{n}} = \frac{1}{n}\mathrm{Vol}(K)^{\frac{1}{n}-1}(\mathrm{P}(K) - n\mathrm{Vol}(K)) \geq \mathrm{Vol}(B_n)^{\frac{1}{n}} - \mathrm{Vol}(K)^{\frac{1}{n}}$$

Therefore, we obtain the Isoperimetric Inequality

$$\mathrm{P}(K) \geq n\mathrm{Vol}(B_n)^{\frac{1}{n}}\mathrm{Vol}(K)^{1-\frac{1}{n}}$$

Now Proposition 2.2.1 implies that if K is a ball then we have equality. Now if we have equality then $f'(0) = f(1) - f(0)$. Since f is concave it follows from Lemma 2.3.3 that f is linear. Thus by corrolary 2.3.2 we have that $K = sB_n + t$ for $s \geq 0$ and $t \in \mathbb{R}$. Which means that if we have equality then f is a ball.

□

3 References

[PG] Penelope Gehring, *The Isoperimetric Inequality: Proofs by Convex and Differential Geometry*, Rose-Hulman Undergraduate Mathematics Journal, Volume 20 (2019)

[AT] Anderjs Treibergs, *Inequalities that Imply the Isoperimetric Inequality*, University of Utah (2002)