WHAT IS COMPLETENESS?

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Abstract. The two, related notions of completeness in model theory are at the core of many beautiful theorems. In this expository note, we exhibit the eponymous Completeness Theorem (without proof) and briefly discuss its high-level meaning— which should be of interest to any mathematician. From this and results about "complete theories," we will then easily deduce nontrivial results in graph theory, algebra, and geometry, all of whose statements contain no obvious influence from logic! These include the De Bruijn-Erdős theorem, the Noether-Ostrowski irreducibility criterion, and the Ax-Grothendieck theorem. To make this accessible to mathematicians without any logic background, we will be spartan with definitions and use blackboxes as-needed.

1. The Completeness Theorem

Many mathematicians without a background in logic are apprehensive of its notation and/or subject matter. However, beginning with the statement of the Completeness Theorem will elucidate exactly which ideas are necessary for us while also introducing today's main character. After stating the theorem, we will immediately unpack the terms in concrete ways and see that it is hardly something to be apprehensive about! Seasoned logicians reading this will have to forgive the informality of some definitions and theorem statements, which will differ from convention in certain areas. That aside, we will be examining the following theorem, first found in Gödel's 1929 PhD thesis [6]:

Theorem 1.1 (Gödel's Completeness Theorem)**.** *For a language $\mathcal{L}$, a theory T, and a sentence φ, there is a proof of φ from the axioms of T iff φ is true in any model of T.*[1]

Let us concisely unpack what this means, before seeing examples of each notion. We first need to formalize the mathematics we are talking about– to do this we introduce languages. A *language* $\mathcal{L}$ is the union of three sets: one of function symbols $\mathcal{F} = \{f_i\}$, one of relation symbols $\mathcal{R} = \{R_i\}$, and one of constant symbols $\mathcal{C} = \{c_i\}$. For a particular $f \in \mathcal{F}$ or $r \in \mathcal{R}$, we call the number of its inputs its *arity*, and we allow relations to have an arity of 0.

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[1]For proofs, one could see the original source [6], but significantly more concise and transparent proofs were given later. Chang and Keisler [4] Chapter 2 gives a full proof.

Further, if $f \in \mathcal{F}$ or $r \in \mathcal{R}$ has arity $n \in \mathbb{Z}_{\geq 0}$, then the domain is $\mathcal{C}^n$ and the codomain is $\mathcal{C}^m$ (for some $m \geq 1$) or $\{0,1\}$, respectfully (the latter can be viewed as 'false or true'). A *sentence* φ is just some 'meaningful,' finite string of symbols from $\mathcal{L}$ and the following collection of symbols:

$$=, \quad \underbrace{\{x_n\}_{n\in\mathbb{N}},}_{\text{variables}} \quad \underbrace{\neg,\ \wedge,\ \vee,\ \implies,}_{\text{logical connectives}} \quad \underbrace{\exists,\ \forall,}_{\text{quantifiers}} \quad \underbrace{(,\ )}_{\text{parentheses}} \quad ,$$

where each variable appearing in φ must be bound (i.e. corresponding) to a quantifier, and 'meaningful' is in the sense of the math you already know from introductory proof classes ('$\forall x_1 : x_1 = x_1$' is meaningful, but '$)x_1\exists\neg c_1$' is not). We often add a colon ':' for notational clarity to say that quantification has ended, but it is not part of the language. Finally, given a language $\mathcal{L}$, a *theory* T is a set of sentences (dependent on $\mathcal{L}$), often called axioms, and a *model* M of T is a set such that $\varphi \in T$ implies φ is true when quantified over M. We say T proves a sentence φ if one can start with the axioms and arrive at φ after a finite number of deductions– deductions simply of the kind you are used to in a proofs class.

For example, one could work in the language of groups: $\mathcal{L}_{grp} := \{\times, \cdot^{-1}\} \cup \{1\}$, where we use multiplicative notation and note that there are no relations. The function $\times$ has arity 2, while the function $\cdot^{-1}$ has arity 1. Easy examples of sentences include the following:

$$\underbrace{\exists x_1 \forall x_2 : x_1 \times x_2 = x_2}_{\text{there exists an identity}} \quad \text{or} \quad \underbrace{\forall x_1, x_2 : x_1 \times x_2 = x_2 \times x_1}_{\text{the group is abelian}} \tag{1}$$

Clearly the first sentence is true for all groups, while the latter is only true for abelian groups– but this is allowed (a 'meaningful' sentence has only to do with the sentence being well-defined, *not* with its truthfulness). Further, the theory of groups is simply the axioms you already know and love– associativity, the existence of an identity, and the properties of inverses! Some models for this theory include $(\mathbb{Z}, +), (\mathbb{C}, \cdot)$, or $(S_3, \cdot)$. Notice that if we appended the second sentence in (1) as an axiom of our theory (thereby forming the theory of *abelian* groups), $(S_3, \cdot)$ is no longer a model of this new theory.

This aside, what is Gödel's Completeness Theorem saying? If one concedes that the goal of mathematics is to find statements that are true, and prove that they are true, then completeness says that this can *always* be done– the goal of mathematics is achievable! The only catch here is that such statements are limited to sentences, as we have defined them, whereas there are certainly mathematical theorems/properties that cannot be expressed this way.[2] Nonetheless, that even this nontrivial fragment of mathematical content is provably

[2] This is the limitation of model theory as given above– one can only quantify over sets (not functions, sets of sets, etc.). Examples of properties that cannot be expressed as sentences include: that a group is torsion, that a graph is connected, or that a ring is a PID. Whats known as "higher-order" logics can account for

within the our grasp is something we often taken for granted. Indeed, many mathematicians equivocate between truth and provability as desired, whereas these notions are connected via Theorem 1.1 in a more nuanced way. If this quasi-philosophical content is not your favorite, fear not– the rest of this note will not worry so much about interpreting Theorem 1.1, but rather focus on applications it to areas of mathematics you likely already know and love!

What Theorem 1.1 implicitly suggests is a very interesting sort of mathematical theory– one where we can equivocate between truth and provability. Thence will we call a theory T *complete* if for all sentences φ written in the same language as T, T proves exactly one of φ or $\neg\varphi$. For example, we already remarked that the theory of groups has models such as $(\mathbb{Z}, +)$ where the sentence

$$\forall x_1, x_2 : x_1 \times x_2 = x_2 \times x_1$$

is true, as well as models such as $(S_3, \cdot)$ where it is false. Hence the theory of groups cannot prove this sentence or its negation and therefore, by Theorem 1.1, this theory is incomplete. As nontrivial examples of complete theories, we give the following theorem which will be needed in the next section:

Theorem 1.2. *The theory of algebraically closed fields of fixed characteristic is complete.*

To be concrete, this means that in the language of fields $\mathcal{L}_{flds} := \{+, -, \times, \cdot^{-1}\} \cup \{0, 1\}$ one can consider the theory of algebraically closed fields of characteristic p, denoted ACF_p, or characteristic 0, denoted ACF_0. These theories are generated by the field axioms along with a few others: for any $n \in \mathbb{Z}_{\geq 1}$, and regardless of characteristic, one adds the axiom

$$\underbrace{\forall y_1, \ldots y_n \exists x : x^n + y_n x^{n-1} + \cdots + y_2 x + y_1 = 0}_{\text{all polynomials of degree } n \text{ have a root}},$$

then for ACF_p one adds $p \cdot 1 = 0$ and for ACF_0 one adds the axioms $\neg(p \cdot 1 = 0)$ for all primes p. There are two different proofs of Theorem 1.2, one model-theoretic due to Tarski and one purely algebraic essentially due to Chevalley, that are both rather elementary and short (< 1 page) using the right ideas.[3] Despite their nature, either proof would lead us astray of our goals, so we elect to simply blackbox Theorem 1.2 for later use.

As a final remark for the philosophically-inclined, the Completeness Theorem does not contradict Gödel's more famous Incompleteness Theorems, which, in colloquial terms, imply

this and have more expressibility, but are beyond our scope and the completeness theorem is generally not true in that context.

[3]See Marker [8] Chapter 3 for the first and EGA IV [7] for the second. Notably, the second proof method by Chevalley did not explicitly aim to prove such a result, but logicians later realized that his work immediately implied something far stronger (quantifier elimination) when translated into a model-theoretic framework. Expositions of this translation abound online.

that there are 'true' statements about the theory of arithmetic in the natural numbers that cannot be proven.[4] Without getting into details, the nuance is that the 'truthfulness' of the sentence given by Gödel's Incompleteness Theorem is dependent on the model of the natural numbers one considers. In the standard model of the natural numbers that you know and love, the sentence is evidently true, whereas there is a model of natural numbers that has nonstandard, infinite elements where the statement is false.[5] Hence Theorem 1.1 implies that such a sentence has no proof from the theory of arithmetic.

2. Applications to Graphs, Algebra, and Geometry.

In this section, we use Theorems 1.1 and 1.2 to arrive at some beautiful mathematical theorems that prima facie have no connection to mathematical logic. The original proofs of these theorems involved each required long page lengths and noteworthy mathematicians to tackle– we will do it with relative ease!

A leitmotif is that applying Theorem 1.1 or 1.2 is very similar to appealing to compactness from elementary topology.[6] In other words, a property expressed by a sentence will often be true if and only if some 'finitistic version' of this property is true for all finite subsets of our structure. The proceeding subsections each give a short lemma to make this precise, after which clever application gives way to graph coloring, polynomial irreducibility, and bijectivity criteria.

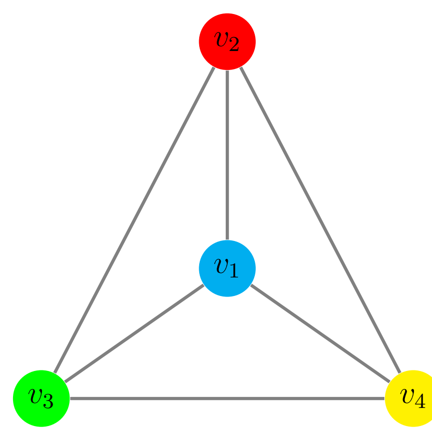

Figure 1. A 4-colorable graph.

2.1. **Graph Colorings.** First, let us consider a (planar) graph G consisting of a set of vertices $V = \{v_n\}$ and a set of edges $E = \{e_n\}$. By a subgraph $H \subseteq G$, we mean some subset of vertices $V' \subseteq V$ along with the edges E' of E that have both vertices in V'. For fixed $k \in \mathbb{Z}_{\geq 1}$, a k-coloring is a way to partition the vertices of the graph into k subsets, such that if two vertices are connected by an edge, they are contained in different subsets. A famous theorem of Appel

[4]Logicians must forgive me the distinction between the theory of arithmetic and the Peano Axioms.

[5]This is related to the talk by Saúl Rodríguez Martín several weeks back on the hyperreals, which can be seen as an analogous nonstandard model of the theory of real closed fields.

[6]This resemblance is not merely superficial. Theorem 1.1 is equivalent to the topological compactness of a particular Stone space defined via the sentences of $\mathcal{L}$. See the expositions by Tao [12] or Pierce [11] for more details.

and Haken (proved in 1976 after 100+ years of failed attempts!) says that any planar graph has a 4-coloring [2]. The following is related:

Theorem 2.1 (De Bruijn–Erdős, 1951; [5])**.** *A graph G has a k-coloring if and only if any finite subgraph $H \subset G$ has a k-coloring.*

To see that this theorem is a corollary of Theorem 1.1, we work with the following, equivalent of Theorem 1.1. We will only prove one direction of this equivalence, since it is all we need. Short as it may be, this will be the most logic-intensive proof of these notes and can be skipped on first reading if desired.

Theorem 2.2 (The Compactness Theorem)**.** *Fix a language $\mathcal{L}$ and a theory T. If every finite subset of T has a model, then so does T.*

Proof. Fix a language $\mathcal{L}$ and theory T. Assuming that T does not have a model, then arguing in a contrapositive fashion, it suffices to find a finite subset $T_0 \subset T$ that does not have a model. First, notice that for any sentence φ, Theorem 1.1 says that the axioms of T must prove φ since it is vacuously true in all models of T. Since this is true of $\neg\varphi$ as well, T proves a contradiction of the form $\varphi \wedge \neg\varphi$.[7] Proofs are finite, so only a finite subset $T_0 \subset T$ is necessary for the proof of this contradiction. Thence, T_0 can have no model, as desired. □

From here, it is very easy to prove Theorem 2.1. We prove it for graphs with countably-many vertices to ease notation, but the proof works for graphs and hypergraphs of any cardinality! Further, this exact proof technique can be applied to a wide variety of other coloring-related questions [13].

Proof of Theorem 2.1. Fix $k \in \mathbb{Z}_{\geq 1}$ and a graph $G = (V, E)$. One can associate the vertices V with $\mathbb{N}$ without loss of generality, if desired. One direction is clear, and we prove the other via contrapositive. Hence, assume G does not have a k-coloring. Our strategy here is to create a theory that uniquely specifies G but falsely claims that it has a k-coloring. Then Theorem 2.2 will do all the heavy lifting!

To this end, introduce a 2-ary relation $R(x, y)$ and axioms such that $R(m, k) = 1$ if there is an edge $e \in E$ connecting vertices m and k, and 0 otherwise. This will uniquely identify our graph G. Next, for all $n \in V$ and $i \in \{1, \ldots, k\}$, introduce 0-ary relations $R_{n,i}$ and the

[7] Logicians may recognize this as one half of the Model Existence Theorem of Henkin, which is also equivalent to the Completeness Theorem. Some reports claim that Henkin's original proof came to him in a dream [1].

following axioms:

$$\underbrace{R_{n,1} = 1 \vee \cdots \vee R_{n,k} = 1}_{\text{All vertices have a color}}$$

$$\underbrace{\forall i, j : (i \neq j \wedge R_{n,i} = 1) \implies R_{n,j} = 0}_{\text{No vertex has two colors}}$$

Recall that 0-ary relations are simply saying something is "true" or "false," so the subscripts above are merely labels. Finally, for exactly the $m, k \in V$ adjoined by an edge $e \in E$, add

$$\underbrace{\forall i : R_{m,i} = 1 \implies R_{k,i} = 0}_{\text{No connected vertices have the same color}}$$

to our axioms. Let T be the theory given by the above axioms for all $n \in V$ and $e \in E$. Since G has no k-coloring, T has no model. Theorem 2.2 then gives a finite subset $T_0 \subset T$ of this theory that also has no model. Since T_0 is finite, it only quantifies over finitely-many vertices $V_0 \subsetneq V$. This naturally gives one a finite subgraph $G_0 \subsetneq G$ formed with the vertices of V_0 and the finitely-many edges of E with both vertices in V_0. Since the axioms corresponding to V_0 have no model, there cannot be a k-coloring of G_0. $\square$

2.2. **Algebra and Geometry.** We continue the applications of model-theoretic completeness in a new direction. So far, we have leveraged model theory and the Completeness Theorem in the form of Theorem 2.2, but we have yet to invoke Theorem 1.2 or the idea of complete theories at all. We will see that such theories often allow for the transference of 'nice' properties between models, or better yet, to models of another, closely-related theory. Such a transference will allow us to prove the following theorems:

Theorem 2.3 (Noether-Ostrowski, 1922/19; [9],[10])**.** *For a field K, let $\overline{K}$ denote an algebraic closure of K. Then $f \in \mathbb{Z}[x_1, \dots, x_n]$ is irreducible over $\overline{\mathbb{Q}}$ if and only if $f \mod p$ is irreducible over $\overline{\mathbb{F}_p}$ for almost every prime p.*

Theorem 2.4 (Ax-Grothendieck, 1968/66; [3],[7])**.** *If a polynomial mapping $f = (f_1, \dots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ is injective, then it is also surjective.*[8]

The completeness-infused lemma we will need is the following to prove these is as follows:

Lemma 2.5 (The Lefschetz Principle)**.** *Let $\mathcal{L}_{flds}$ be the language of fields and φ be a sentence. The following are equivalent:*

(1) *φ is true in some algebraically closed field of characteristic* 0,

[8]One can use the same proof methods here to generalize this significantly to endomorphisms of algebraic varieties over algebraically closed fields, for the geometrically inclined.

(2) *φ is true in all algebraically closed fields of characteristic* 0,
(3) *φ is true in some algebraically closed fields of characteristic p, for all large p,*
(4) *φ is true in all algebraically closed fields of characteristic p, for all large p.*

Proof. Notice that (1) $\iff$ (2) and (3) $\iff$ (4) are both immediate from Theorems 1.1 and 1.2. This is the upshot of having a complete theory- since there is a proof of exactly one of φ or $\neg\varphi$, if φ is true in one model, it must be that φ is true for all models.

Hence it suffices to prove (2) $\iff$ (4). Suppose φ is true in all algebraically closed fields of characteristic 0. Theorem 1.1 then gives that ACF_0 proves φ. Since proofs are finite, only finitely-many of the axioms $\neg(p \cdot 1 = 0)$ are needed to prove φ, so φ is provable with ACF_p for all large p. By Theorem 1.1, this gives (4). The mapping $\varphi \mapsto \neg\varphi$ gives the converse. $\square$

Proof of Theorem 2.3. Fix $f \in \mathbb{Z}[x_1, \dots, x_n]$. Observe that f is irreducible if and only if it cannot be written as a product of two polynomial of degree less than $\deg f := k$ whose degrees add to k. Hence the irreducibility of f is expressed by the sentence

$$\forall a_1, \dots, a_{2k} : (a_{2k}x^k + \cdots a_{k+2}x + a_{k+1})(a_kx^k + \cdots a_2x + a_1) = f$$
$$\implies (a_{2k} = \cdots = a_{k+2} = 0 \vee a_k = \cdots = a_2 = 0)$$

Lemma 2.5 then immediately gives the desired conclusion. $\square$

Proof of Theorem 2.4. Let $n, k \in \mathbb{Z}_{\geq 1}$ be arbitrary and fixed. It is easy, albeit tedious, to see that the statement "every injective polynomial map in n coordinates with degree $\leq k$ is surjective" can be written as a sentence and we leave it as an exercise. Call this sentence $\varphi_{n,k}$. By Lemma 2.5, if we prove that $\varphi_{n,k}$ holds over $\overline{\mathbb{F}_p}$ for all large p, we are done.

Hence, for p prime, suppose that $f : \overline{\mathbb{F}_p}^n \to \overline{\mathbb{F}_p}^n$ is injective with $\deg f \leq k$. A result from undergraduate algebra says that $\overline{\mathbb{F}_p} = \bigcup_{\ell \in \mathbb{Z}_{\geq 1}} \mathbb{F}_{p^\ell}$. Since f has only finitely many coefficients, this implies that $f \in \mathbb{F}_{p^m}[x_1, \dots, x_n]$ for some $m \in \mathbb{Z}_{\geq 1}$. Hence for $\ell \geq m$, the restriction $f : \mathbb{F}_{p^\ell} \to \mathbb{F}_{p^\ell}$ is injective, and since $\mathbb{F}_{p^\ell}$ is finite, this restriction is automatically surjective. Since $\ell \geq m$ was arbitrary, the statement remains true for $\overline{\mathbb{F}_p} = \bigcup_{\ell \in \mathbb{Z}_{\geq 1}} \mathbb{F}_{p^\ell}$, as desired.[9] $\square$

[9]It is of note that $\varphi_{n,k}$ has quantifiers of the form $\forall x_1, \dots, x_m \exists x_{m+1}, \dots, x_{m+p}$, and any sentence of this kind satisfies something stronger than Lemma 2.5– we leave this as a challenge to the reader. Showing this would effectively eliminate the second paragraph of the proof of Theorem 2.4.

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