What is Sharkovsky's Theorem, or why does period 3 imply chaos?

Dhruv Goel

Def 1. A **discrete dynamical system (DDS)** is a pair (S, f), where S is a set & $f: S \to S$ a function.
For each $n \geq 0$, define **n^{th} iterate f^n of f** by $f^0 = id_S$ & $f^{n+1} = f \circ f^n$ for $n \geq 0$.
Given $x \in S$, define the **forward orbit O_x of x** by $O_x := \{f^n x : n \geq 0\} \subseteq S$,
and the set **$Per(x)$ of periods of x** by $Per(x) := \{n : f^n x = x\} \subseteq \mathbb{Z}_{\geq 1}$.

We say that $x \in S$ is

i) **periodic** iff $Per(x) \neq \emptyset$. The **least period of** x is $LP(x) := \min Per(x)$.
If $LP(x) = n \geq 1$, then $Per(x) = n\mathbb{Z}_{\geq 1}$.

ii) **preperiodic** iff O_x is finite $\Leftrightarrow$ $\exists n \geq 0$ s.t. f^n is periodic.

iii) **wandering** iff O_x is infinite.

Define the **set of least periods** to be $LP(f) := \bigcup_{\substack{x \in S \\ \text{periodic}}} LP(x) \subseteq \mathbb{Z}_{\geq 1}$.

Goal: Given DDS, Study iterates f^n & $LP(f)$.

eg. $LP(f) \neq \emptyset \Leftrightarrow$ there is a periodic pt

$1 \in LP(f) \leftrightarrow f$ has a fixed pt etc.

Rmk 2. Usually, a DDS (S, f) has more structure

eg. S is a top. space & f cts or
S is a measure space & f measure preserving or
for some Category $\mathcal{C}$, we have $S \in Ob(\mathcal{C})$ & $f \in End_{\mathcal{C}}(S) = Mor_{\mathcal{C}}(S, S)$.

Now: focus on $S = [0,1]$ & $f : [0,1] \longrightarrow [0,1]$ continuous (= cts.)

Q. What are possible $LP(f)$ for cts $f : [0,1] \to [0,1]$?

Eg. Intermediate Value Thm (IVT) $\Rightarrow 1 \in LP(f)$. (See below.)

Convention 3. An **interval** is a nonempty closed interval $\subset \mathbb{R}$, i.e. a set $[a,b] = \{x : a \le x \le b\} \subset \mathbb{R}$ for $a, b \in \mathbb{R}$ & $a \le b$. In particular, the singleton $\{x\} = [x, x]$ for $x \in \mathbb{R}$ is allowed.

Example 4.

① $f(x) = x$. Every x is a fixed pt & $LP(f) = \{1\}$.

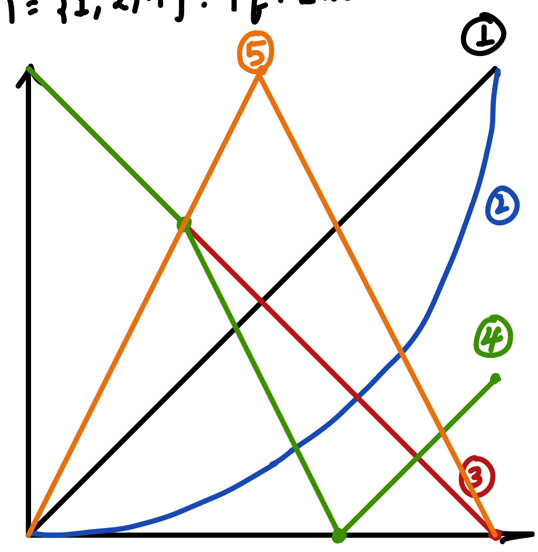
② $f(x) = x^2$. For $x \in [0,1]$, x is per. $\Leftrightarrow$ preper. $\Leftrightarrow$ fixed $\Leftrightarrow x \in \{0,1\}$.
In this case, $LP(f) = \{1\}$.

Similarly, $f(x) = x^m$, $m \in \mathbb{Z}_{\ge 2}$.

③ $f(x) = 1 - x$. Every pt is periodic with least period 2, except $x = 1/2$, which is a fixed pt $\therefore LP(f) = \{1, 2\}$.

④ $f(x) = \begin{cases} 1 - x, & x \in [0, 1/3] \\ 4/3 - 2x, & x \in [1/3, 2/3] \\ x - 2/3, & x \in [2/3, 1]. \end{cases}$ Then $LP(f) = \{1, 2, 4\}$. Pf. Exercise.

⑤ $f(x) = \begin{cases} 2x, & x \in [0, 1/2] \\ 2 - 2x, & x \in [1/2, 1]. \end{cases}$



Then $\begin{matrix} & \nearrow & 4/9 \\ & & \downarrow \\ 2/9 & \leftarrow & 8/9 \end{matrix}$ $\therefore 3 \in LP(f)$.

In fact, $LP(f) = \mathbb{Z}_{\ge 1}$ & the set of all wandering pts $\subset [0,1]$ is uncountable & dense: chaos!

Thm 5. (Sharkovsky, 1964-65) If $f:[0,1]\to[0,1]$ is cts, then

$$3\in LP(f) \Rightarrow LP(f)=\mathbb{Z}_{\geq 1}.$$

Rmk 6. This is not true in other contexts, eg. if $S=[0,1]^2$ or $S=S^1$, eg. $z\mapsto e^{2\pi i/3}$.

The key input is the Intermediate Value Thm:

Thm 7. (Intermediate Value Thm / IVT) $c\in[\min\{f(a),f(b)\}, \max\{f(a),f(b)\}]$

If $f:[a,b]\to\mathbb{R}$ is cts & $c\in\mathbb{R}$ between $f(a)$ & $f(b)$, then $\exists\, x\in[a,b]$ st $f(x)=c$.

Pf of Thm 5. Let $f:[0,1]\to[0,1]$ be cts.

Step I. If $I, I'\subseteq[0,1]$ intervals st $f(I)\supseteq I'$, then $\exists$ interval

$$J\subseteq I \text{ st } f(J)=I.$$

Pf. Let $I'=[c,d]$. If $c=d$, done. Else, $c<d$.

Pick $a,b\in I$ st $f(a)=c$ & $f(b)=d$.

Suppose $a<b$; other case is similar.

Then $p:=\sup\left(f^{-1}(c)\cap[a,b]\right)\in f^{-1}(c)\cap[a,b)$,

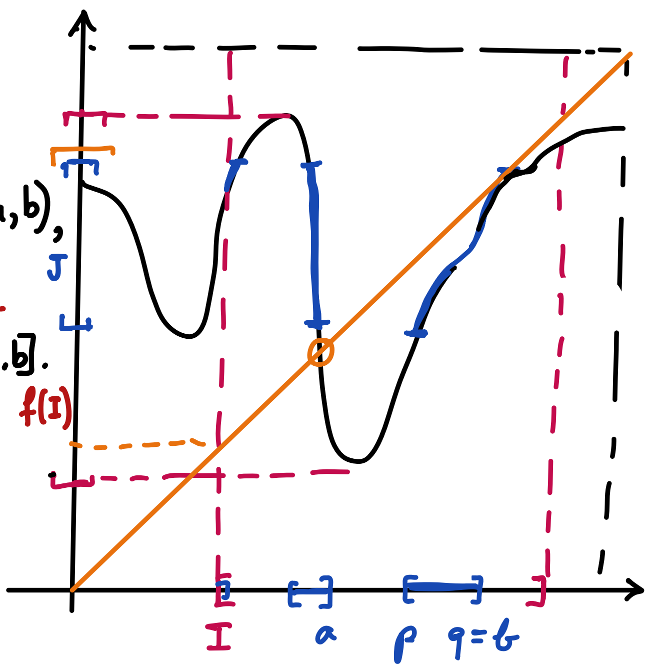
closed, bounded, nonempty.

and $q:=\inf\left(f^{-1}(d)\cap[p,b]\right)\in f^{-1}(d)\cap(p,b]$.

Claim: $J=[p,q]$ works.

Well, $f(J)\supseteq I'$ by IVT (Thm 7).

If $\exists\, y\in J$ st $f(y)\notin I'$, then either $f(y)<c$ or $f(y)>d$.



If $f(y)<c$, then $y>p$ & by IVT, $\exists\, p'\in[y,b]$ st $f(p')=c$, contradiction to choice of p.

If $f(y)>d$, then $y<q$ & by IVT, $\exists\, q'\in[p,y]$ st $f(q')=d$, contradiction to choice of q.

Similarly, if $a>b$, take $p:=\sup f^{-1}(d)\cap[b,a]$ & $q:=\inf f^{-1}(c)\cap[p,a]$. ▨

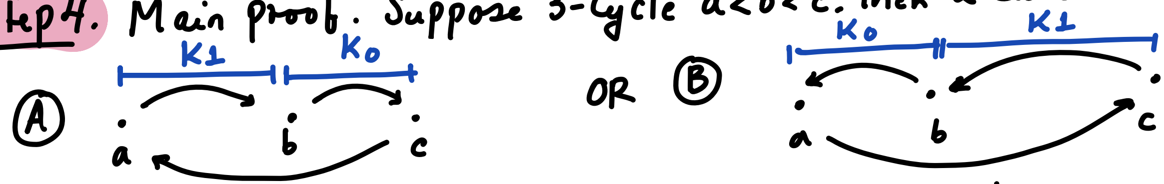
Step 2. If $I \subseteq [0,1]$ interval s.t. $f(I) \supseteq I$, then $\exists\, x \in I$ st $f(x) = x$.

Pf. Say $I = [c,d]$ & let $a, b \in I$ st $f(a) = c$ & $f(b) = d$. Then
$f(a) - a \leq 0 \leq f(b) - b$, so by IVT applied to $g(x) = f(x) - x$. ▨

Step 3. For any integer $n \geq 1$, if $I_0, \dots, I_{n-1} \subseteq [0,1]$ intervals s.t. if $I_n := I_0$ then for all $j = 0, 1, \dots, n-1$ have $f(I_j) \supseteq I_{j+1}$,
then $\exists\, x \in I$ st $f^n(x) = x$ and $f^j(x) \in I_j$ for $j = 0, 1, \dots, n-1$.

Pf. Set $J_n := I_0$. By step 1, $\exists\, J_{n-1} \subseteq I_{n-1}$ st $f(J_{n-1}) = J_n$. Then
$\exists\, J_{n-2} \subseteq I_{n-1}$ st $f(J_{n-2}) = J_{n-1}$. Inductively, $\exists$ intervals
$J_0, \dots, J_{n-1} \subseteq [0,1]$ st $\forall\, j \in \{0, 1, \dots, n-1\}$, have $f(J_j) = J_{j+1}$.
Then $\forall\, j \in \{0, \dots, n\}$, we have $f^j(J_0) = J_j$ $\therefore f^n(J_0) = J_n = I_0 \supseteq J_0$.
$\therefore$ By step 2, $\exists\, x \in J_0$ st $f^n(x) = x$. Then also $f^j(x) \in f^j(J_0) = J_j \subseteq I_j$ $\forall\, j \in \{0, \dots, n-1\}$. ▨

Step 4. Main proof. Suppose 3-cycle $a < b < c$. Then 2 cases



Let K_0, K_1 be intervals as indicated, so by IVT (Thm 7) we have
$$f(K_0) \supseteq [a,c] \supseteq K_0, K_1 \text{ and } f(K_1) \supseteq K_0.$$
We know $1 \in LP(f)$ by Step 2 applied to K_0.

For $n = 2$, take $I_0 = K_0$ & $I_1 = K_1$. By step 3, $\exists\, x \in K_0$ st $f^2(x) = x$.
If $f(x) = x$, then $x \in K_0 \cap K_1 = \{b\}$, but then $c = f(b) \neq b$, Contradiction.
Therefore, $f(x) \neq x$ & so $2 \in LP(f)$.

The case $n = 3$ is given. For $n \geq 4$, we will produce $x \in I$ st $LP(x) = n$.

Take $I_0 = I_1 = \cdots = I_{n-2} = K_0$ & $I_{n-1} = K_1$. By step 3, $\exists\, x \in K_0$ st $f^n(x) = x$ and $x, f(x), \ldots, f^{n-2}(x) \in K_0$ while $f^{n-1}(x) \in K_1$.

Claim: $LP(x) = n$.

Pf. If not, $\exists\, k : 1 \le k \le n-1$ & $f^k(x) = x$. Then

$$f^{n-1}(x) = f^{n-k-1} \circ f^k(x) = f^{n-k-1}(x) \in K_0 \cap K_1 = \{b\}$$

In (A), get $x = f \circ f^{n-1}(x) = f(b) = c$ & then $f(x) = a \notin K_0$, contradiction.

In (B), get $x = f(b) = a$ & then $f(x) = c \notin K_0$, a contradiction. $\blacksquare$

In fact, there is a complete answer to the motivating question.

Def 8. The **Sharkovsky order** is the total order on $\mathbb{Z}_{\geq 1}$ given as

$3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2\cdot 3 \rhd 2\cdot 5 \rhd 2\cdot 7 \rhd \cdots \rhd 2^2\cdot 3 \rhd 2^2\cdot 5 \rhd 2^2\cdot 7 \rhd \cdots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1$.

A **tail** of the Sharkovsky order is a nonempty subset $T \subset \mathbb{Z}_{\geq 1}$ s.t. if $a, b \in \mathbb{Z}_{\geq 1}$, then $a \in T$ and $a \rhd b \Rightarrow b \in T$.

Oleksandr Mykolayovich

Thm 5' (Sharkovsky, 1964-65). If $f : [0,1] \to [0,1]$ is cts., then $LP(f)$ is a tail of the Sharkovsky order.

Conversely, if $T \subset \mathbb{Z}_{\geq 1}$ is a tail of the Sharkovsky order, then $\exists$ cts. $f : [0,1] \to [0,1]$ st $T = LP(f)$.

Tien-Yien, James A.

Thm 9. (Li-Yorke, 1975) If $f : [0,1] \to [0,1]$ cts & $3 \in LP(f)$ then f is **chaotic** i.e. $LP(f) = \mathbb{Z}_{\geq 1}$ and the set $W \subset [0,1]$ of wandering pts is uncountable & dense.

History:

a) Coppel, 1955. If $LP \supsetneq \{1\}$, then $2 \in LP(f)$.

b) Sharkovsky, '64-65. (Uk. Math. Journal)

c) Li-Yorke, '75. (American Math. Monthly).

d) Yorke attended a Conference in East Berlin & during a Cruise, a Ukrainian participant approached him, who managed to convey (with the help of translation) that he had proved it already. This was Sharkovsky.

Li-Yorke's article introduced the notion of "chaos" and eventually led to global recognition of Sharkovsky's work.

Source:

Burns-Hasselbatt, "The Sharkovsky Theorem: A Natural Direct Proof." The American Math. Monthly, Vol. 118, 2011, Issue 3, pp. 229-244.

Also available at

https://math.arizona.edu/~dwang/BurnsHasselblattRevised-1.pdf