What is Sharkovsky's Theorem, or why does period 3 imply chaos?

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**Definition 1.** A discrete dynamical system (DDS) is a pair \( (S, f) \), where \( S \) is a set of \( S \times S \).

For each \( n \geq 0 \), define the \( n \)-th iterate \( f^n \) of \( f \) by \( f^0 = \text{ids} \) and \( f^n = f \circ f^{n-1} \).

Given \( x \in S \), define the forward orbit \( O_x \) of \( x \) by \( O_x := \{ f^n(x) : n \geq 0 \} \subseteq S \), and the set \( \text{Per}(x) \) of periods of \( x \) by \( \text{Per}(x) := \{ n : f^n(x) = x \} \subseteq \mathbb{Z}_+ \).

We say that \( x \in S \) is

i) **periodic** iff \( \text{Per}(x) \neq \emptyset \). The least period of \( x \) is \( \text{LP}(x) := \min \text{Per}(x) \).

   If \( \text{LP}(x) = n \geq 1 \), then \( \text{Per}(x) = n \mathbb{Z}_+ \).

ii) **preperiodic** iff \( O_x \) is finite \( \Leftrightarrow \exists n > 0 \) s.t. \( f^n \) is periodic.

iii) **wandering** iff \( O_x \) is infinite.

Define the set of least periods to be \( \text{LP}(f) := \bigcup_{x \in S} \text{LP}(x) \subseteq \mathbb{Z}_+ \).

**Goal:** Given DDS, study iterates \( f^n \) & \( \text{LP}(f) \).

eg. \( \text{LP}(f) \neq \emptyset \Leftrightarrow \) there is a periodic pt.

1 \( \in \text{LP}(f) \Leftrightarrow f \) has a fixed pt etc.

**Remark 2.** Usually, a DDS \( (S, f) \) has more structure

eg. \( S \) is a top. space & \( f \) cont or

\( S \) is a measure space & \( f \) measure preserving or

for some Category \( C \), we have \( S \in \text{Ob}(C) \) & \( f \in \text{End}_C(S) = \text{Mor}(S, S) \).
Now: focus on $S = [0, 1] \times f: [0, 1] \rightarrow [0, 1]$ continuous (= cts.)

Q. What are possible $\text{LP}(f)$ for $f: [0, 1] \rightarrow [0, 1]$?

Eg. Intermediate Value Thm (IVT) $\Rightarrow \exists \in \text{LP}(f)$. (See below.)

**Convention 3.** An interval is a nonempty closed interval $c \mathbb{R}$, i.e. a set $[a, b] = \{x: a \leq x \leq b\} \subset \mathbb{R}$ for $a, b \in \mathbb{R}$ and $a \leq b$. In particular, the singleton $\{x\} = [x, x]$ for $x \in \mathbb{R}$ is allowed.

**Example 4.**

1. $f(x) = x$. Every $x$ is a fixed pt $\Rightarrow \text{LP}(f) = \{1\}$.
2. $f(x) = x^2$. For $x \in [0, 1]$, $x$ is per. $\iff$ prep. $\iff$ fixed $\iff x \in \{0, 1\}$.
   - In this case, $\text{LP}(f) = \{1\}$.

   Similarly, $f(x) = x^m$, $m \in \mathbb{Z} \geq 2$.
3. $f(x) = 1-x$. Every pt is periodic with least period $2$, except $x = \frac{1}{2}$, which is a fixed pt $\Rightarrow \text{LP}(f) = \{1, 2\}$.
4. $f(x) = \begin{cases} 1-x, & x \in [0, \frac{1}{3}] \\ \frac{1}{3} - 2x, & x \in [\frac{1}{3}, \frac{1}{3}] \\ x - \frac{2}{3}, & x \in [\frac{2}{3}, 1]. \end{cases}$
   - Then $\text{LP}(f) = \{1, 2, 4\}$. Pf. Exercise.
5. $f(x) = \begin{cases} 2x, & x \in [0, \frac{1}{2}] \\ 2 - 2x, & x \in [\frac{1}{2}, 1]. \end{cases}$
   - Then $\frac{4}{9} \rightarrow \frac{1}{2} \Rightarrow 3 \in \text{LP}(f)$.

In fact, $\text{LP}(f) = \mathbb{Z} \geq 1$ is the set of all wandering pts $c [0, 1]$ is uncountable & dense: chaos!
**Thm 5. (Sharkovsky, 1964-65)** If \( f : [0,1] \to [0,1] \) is cts, then
\[
3 \in \text{LP}(f) \implies \text{LP}(f) = \mathbb{Z} \cup \{1\}.
\]

**Remark:** This is not true in other contexts, e.g., if \( S = [a,b]^2 \) or \( S = S^2 \).

The key input is the Intermediate Value Thm:
\[
c \in [\min\{f(a), f(b)\}, \max\{f(a), f(b)\}]
\]

**Thm 7. (Intermediate Value Thm / IVT)**
If \( f : [a,b] \to \mathbb{R} \) is cts & \( c \in \mathbb{R} \) between \( f(a) \) & \( f(b) \), then \( \exists x \in [a,b] \) st \( f(x) = c \).

*Proof of Thm 5. Let \( f : [0,1] \to [0,1] \) be cts.
* Skip if \( I \), \( I' \subseteq [0,1] \) intervals st \( f(I) \cap I' = \emptyset \), then \( \exists \) interval \( J \subseteq I' \) s.t. \( f(J) = I' \).

If \( c = a \), done. Else, \( c < d \).

Pick \( a, b \in I \) st \( f(a) = c \) & \( f(b) = d \).

Suppose \( a < b \); other case is similar.

Then \( p := \sup (f^{-1}(c) \cap [a,b]) = f^{-1}(c) \cap [a,b] \),
\( \rightarrow \) closed, bounded, nonempty.

and \( q := \inf (f^{-1}(d) \cap [p,b]) = f^{-1}(d) \cap [p,b] \).

*Claim:* \( J = [p, q] \) works.

Well, \( f(J) \supseteq I' \) by IVT (Thm 7).

If \( \exists y \in J \) st \( f(y) \notin I' \), then either \( f(y) < c \) or \( f(y) > d \).

If \( f(y) < c \), then \( y < p \) by IVT, \( \exists p' \in [y, b] \) st \( f(p') = c \), contradiction.

Choice of \( p \).

If \( f(y) > d \), then \( y < q \) by IVT, \( \exists q' \in [p, y] \) st \( f(q') = d \), contradiction.

Choice of \( q \).

Similarly, if \( a > b \), take \( p := \sup f^{-1}(d) \cap [b, a] \) & \( q := \inf f^{-1}(c) \cap [p,a] \).
**Step 2.** If $I \subseteq [0,1]$ interval s.t. $f(I) \supseteq I$, then $\exists x \in I$ s.t. $f(x) = x$.

Pf. Say $I = [c,d]$ & let $a,b \in I$ s.t. $f(a) = c \land f(b) = d$. Then $f(b) - b \leq f(a) - a \leq 0$, so by IVT applied to $g(x) = f(x) - x$.

**Step 3.** For any integer $n \geq 1$, if $I_0, \ldots, I_{n-1} \subseteq [0,1]$ intervals s.t. $f(I_n) = I_n$ then for all $j = 0, 1, \ldots, n-1$ have $f(I_j) \supseteq I_{j+1}$, then $\exists x \in I$ s.t. $f^n(x) = x$ and $f^j(x) \in I_j$ for $j = 0, 1, \ldots, n-1$.

Pf. Set $J_n := I_0$. By step 1, $\exists J_{n-1} \subseteq I_{n-1}$ s.t. $f(J_{n-1}) = J_n$. Then $\exists J_{n-2} \subseteq I_{n-1}$ s.t. $f(J_{n-2}) = J_{n-1}$. Inductively, $\exists$ intervals $J_0, \ldots, J_n \subseteq [0,1]$ s.t. $\forall j \in \{0, 1, \ldots, n-1\}$, have $f(J_j) = J_{j+1}$. Then $\forall j \in \{0, 1, \ldots, n\}$, we have $f^j(J_0) = J_j \implies f^n(J_0) = J_n = I_0 \supseteq J_0$.

By step 2, $\exists x \in J_0$ s.t. $f^n(x) = x$. Then also $f^j(x) \in f^j(J_0) = J_j \subseteq I_j$ $\forall j \in \{0, 1, \ldots, n\}$.

**Step 4.** Main Proof. Suppose 3-Cycle $a < b < c$. Then 2 Cases

A) \[
\begin{array}{ccc}
K_1 & \downarrow & K_0 \\
\circ & \circ & \circ \\
a & \rightarrow & b \\
& \rightarrow & c
\end{array}
\] OR B) \[
\begin{array}{ccc}
K_1 & \downarrow & K_0 \\
\circ & \circ & \circ \\
a & \rightarrow & b \\
& \rightarrow & c
\end{array}
\]

Let $K_0, K_1$ be intervals as indicated, so by IVT (Thm 7) we have $f(K_0) \supseteq [a, c] \supseteq K_0, K_1$ and $f(K_1) \supseteq K_0$.

We know $1 \in LP(f)$ by step 2 applied to $K_0$.

For $n = 2$, take $I_0 = K_0$ & $I_1 = K_1$. By step 3, $\exists x \in K_0$ s.t. $f(x) = x$.

If $f(x) = x$, then $x \in K_0 \cap K_1 = \{b\}'s$, but then $c = f(b) \neq b$, contradiction.

Therefore, $f(x) \neq x$ s.t. $2 \in LP(f)$.

The case $n = 3$ is given. For $n > 4$, we will produce $x \in I$ s.t. $LP(x) = n$. 


Take $I_0 = I_1 = \cdots = I_{n-2} = K_0 \cap I_{n-1} = K_1$. By step 3, \( \exists x \in K_0 \) s.t.
\[ f^n(x) = x \text{ and } x, f(x), \ldots, f^{n-1}(x) \in K_0 \text{ while } f^{n-1}(x) \in K_1. \]

Claim: \( LP(x) = n \).

\textbf{Pf.} If not, \( \exists k : 1 \leq k \leq n-1 \) \( f^k(x) = x \). Then
\[ f^{n-1}(x) = f^{n-k-1} f^k(x) = f^{n-k-1}(x) \in K_0 \cap K_1 = \{ b \} \]

In (A), get \( x = f \) \( f^{n-1}(x) = f(b) = c \) \( \text{ then } f(x) = a \notin K_0 \), contradiction.

In (B), get \( x = f(b) = a \) \( \text{ then } f(x) = c \notin K_0 \), a contradiction. \( \blacksquare \)

In fact, there is a complete answer to the motivating question.

**Def 8.** The \textbf{Sharkovsky order} is the total order on \( \mathbb{Z}_{\geq 1} \) given as
\[ 3 > 5 > 7 > \cdots > 2 \cdot 3 > 2 \cdot 5 > 2 \cdot 7 > \cdots > 2^3 > 2^5 > 2^7 > \cdots > 2^{10} > 2^{20} > \cdots. \]

A \textbf{tail} of the Sharkovsky order is a nonempty subset \( T \subseteq \mathbb{Z}_{\geq 1} \) s.t.
if \( a, b \in \mathbb{Z}_{\geq 1} \), then \( a \in T \text{ and } a \nless b \Rightarrow b \in T. \)

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\text{Oleksandr Myklayovich}
\end{center}

**Thm 5' (Sharkovsky, 1964-65).** If \( f : [0,1] \to [0,1] \) is cts., then \( LP(f) \) is a tail of the Sharkovsky order.

Conversely, if \( T \subseteq \mathbb{Z}_{\geq 1} \) is a tail of the Sharkovsky order, then \( \exists \text{ cts. } f : [0,1] \to [0,1] \text{ s.t. } T = LP(f). \)

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**Thm 9.** (Li-Yorke, 1975) If \( f : [0,1] \to [0,1] \) cts. and \( 3 \in LP(f) \)
then \( f \) is \textbf{chaotic} i.e. \( LP(f) = \mathbb{Z}_{\geq 1} \) and the set \( W \subseteq [0,1] \) of wandering pts is uncountable & dense.
History:

a) Coppel, 1955. If \( LP \supset f \), then \( 2 \in LP(f) \).
b) Sharkovsky, '64-65. (Uk. Math. Journal)
d) Yorke attended a conference in East Berlin & during a cruise, a Ukrainian participant approached him, who managed to convey (with the help of translation) that he had proved it already. This was Sharkovsky.

Lit Yorke's article introduced the notion of "chaos" and eventually led to global recognition of Sharkovsky's work.

Source:


Also available at:

https://math.arizona.edu/~dwang/BurnsHasselblattRevised-1.pdf