2023 Algebra 6111 Qualifying Exam

Instructions

Enter your name.# on the roster sheet together with a code name for yourself that is different from any code name that has already been entered.

Answer each question on a separate sheet of paper, and write your code name and the problem number on each sheet of paper that you submit for grading. Do not put your real name on any sheet of paper that you submit for grading.

Answer as many questions as you can. Do not use theorems which make the solution to the problem trivial. Always clearly display your reasoning. The judgment that you use in this respect is an important part of the exam.

This is a closed book, closed notes exam.

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1. Let H be a proper normal subgroup of a finite group G and let p be a prime factor of $|G/H|$. Show that the number of Sylow p-subgroups of G/H divides the number of Sylow p-subgroups of G.

2. Let G be a finite group such that any two of its proper maximal subgroups are conjugate. Prove that G is cyclic.

3. Let R be a commutative ring and $f_1, f_2, \ldots, f_r \in R$ such that they generate the unit ideal. Show that an R-module M is finitely-generated if and only if for all $i = 1, \ldots, r$ the localization M_{f_i} is a finitely-generated R_{f_i}-module. (Here R_{f_i} and M_{f_i} are the localizations $S^{-1}R$ and $S^{-1}M$ with respect to the multiplicative subset $S = \{f_i^n\}_{n \geq 0}$.)

4. Let D be a unique factorization (commutative) domain, $f(x) = a_0 + a_1x + \cdots + a_nx^n \in D[x]$, $f(x) \neq 0$. The content $c(f)$ of $f(x)$ is defined to be $\gcd(a_0, a_1, \ldots, a_n)$, and $f(x)$ is called primitive if $c(f)$ is associate to 1 (i.e., is a unit of D). Prove the Gauss lemma: if $f(x), g(x) \in D[x]$ are primitive, then $h(x) = f(x)g(x)$ is primitive.

5. Let A be an $n \times n$ matrix of rank k over a field. What is the rank of $\operatorname{adj}(A)$, the adjoint of A? Recall that $\operatorname{adj}(A)$ is the $n \times n$ matrix whose (i, j)-entry is $(-1)^{i+j}$ times the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by removing the jth row and ith column.