**2025 Gordon examination problems**

**1.** Let $F_1, F_2, \dots$ be the sequence of Fibonacci numbers: $F_1 = F_2 = 1$, $F_n = F_{n-2} + F_{n-1}$ for all $n \geq 3$. Find all $n \in \mathbb{N}$ for which the polynomial $F_n x^{n+1} + F_{n+1} x^n - 1$ is irreducible in $\mathbb{Q}[x]$.

**2.** Prove: $\binom{2025}{0} - \binom{2025}{2} + \binom{2025}{4} - \cdots + \binom{2025}{2024} = 2^{1012}$.

**3.** If $a_1, b_1, \dots, a_n, b_n > 0$, prove that $\sqrt[n]{(a_1 + b_1) \cdots (a_n + b_n)} \geq \sqrt[n]{a_1 \cdots a_n} + \sqrt[n]{b_1 \cdots b_n}$.

**4.** Find the sum $\sum_{n=1}^{\infty} \arctan \frac{1}{1+n+n^2}$.

**5.** Each point of $\mathbb{R}^3$ is colored in one of four colors. Prove that there exist two points of the same color that are at distance 1 from each other.

**6.** Find all symmetric $2025 \times 2025$ matrices all of whose entries are either 0 or 1 and such that all their eigenvalues are positive real numbers.