Solutions to 2025 Gordon examination problems

1. *Let $F_1, F_2, \dots$ be the sequence of Fibonacci numbers: $F_1 = F_2 = 1$, $F_n = F_{n-2} + F_{n-1}$ for all $n \geq 3$. Find all $n \in \mathbb{N}$ for which the polynomial $F_n x^{n+1} + F_{n+1} x^n - 1$ is irreducible in $\mathbb{Q}[x]$.*

Solution. Let $P_n(x) = F_n x^{n+1} + F_{n+1} x^n - 1$, $n \in \mathbb{N}$, and $P_0(x) = 0$. The polynomial $P_1(x) = x^2 + x + 1$ is irreducible. By the definition of Fibonacci's numbers, for every $n \geq 2$ we have

$$\begin{aligned}
P_n(x) = F_n x^{n+1} + F_{n+1} x^n - 1 &= (F_{n-1} + F_{n-2}) x^{n+1} + (F_n + F_{n-1}) x^n - 1 \\
&= F_{n-1} x^{n+1} + F_{n-2} x^{n+1} + F_n x^n + F_{n-1} x^n - 1 \\
&= (F_{n-1} x^n + F_n x^{n-1} - 1) x + (F_{n-2} x^{n-1} + F_{n-1} x^n - 1) x^2 + x^2 + x - 1 \\
&= x P_{n-2}(x) + x^2 P_{n-1}(x) + 1.
\end{aligned}$$

By induction, we see that for all n, P_n is divisible by $x^2 + x - 1$, and so, reducible for all $n \geq 2$.

2. *Prove: $\binom{2025}{0} - \binom{2025}{2} + \binom{2025}{4} - \cdots + \binom{2025}{2024} = 2^{1012}$.*

Solution. The sum $S = \binom{2025}{0} - \binom{2025}{2} + \binom{2025}{4} - \cdots + \binom{2025}{2024}$ is the real part of the binomial expansion

$$(1+i)^{2025} = \sum_{n=0}^{2025} \binom{2025}{i} i^n = \binom{2025}{0} + \binom{2025}{1} i - \binom{2025}{2} - \binom{2025}{3} i + \binom{2025}{4} + \binom{2025}{5} i - \cdots + \binom{2025}{2024}.$$

Since $1 + i = \sqrt{2} e^{2\pi i/8}$, $(1+i)^{2025} = \sqrt{2}^{2024}(1+i)$, whose real part is $\sqrt{2}^{2024} = 2^{1012}$.

3. *If $a_1, b_1, \dots, a_n, b_n > 0$, prove that $\sqrt[n]{(a_1 + b_1) \cdots (a_n + b_n)} \geq \sqrt[n]{a_1 \cdots a_n} + \sqrt[n]{b_1 \cdots b_n}$.*

Solution. The inequality is equivalent to

$$\sqrt[n]{\frac{a_1}{a_1 + b_1} \cdot \frac{a_2}{a_2 + b_2} \cdots \frac{a_n}{a_n + b_n}} + \sqrt[n]{\frac{b_1}{a_1 + b_1} \cdot \frac{b_2}{a_2 + b_2} \cdots \frac{b_n}{a_n + b_n}} \leq 1.$$

By the (twice applied) AGM inequality,

$$\begin{aligned}
&\sqrt[n]{\frac{a_1}{a_1 + b_1} \cdot \frac{a_2}{a_2 + b_2} \cdots \frac{a_n}{a_n + b_n}} + \sqrt[n]{\frac{b_1}{a_1 + b_1} \cdot \frac{b_2}{a_2 + b_2} \cdots \frac{b_n}{a_n + b_n}} \\
&\qquad \leq \frac{1}{n}\Big(\frac{a_1}{a_1 + b_1} + \frac{a_2}{a_2 + b_2} + \cdots + \frac{a_n}{a_n + b_n}\Big) + \frac{1}{n}\Big(\frac{b_1}{a_1 + b_1} + \frac{b_2}{a_2 + b_2} + \cdots + \frac{b_n}{a_n + b_n}\Big) \\
&\qquad = \frac{1}{n}\Big(\frac{a_1 + b_1}{a_1 + b_1} + \frac{a_2 + b_2}{a_2 + b_2} + \cdots + \frac{a_n + b_n}{a_n + b_n}\Big) = 1.
\end{aligned}$$

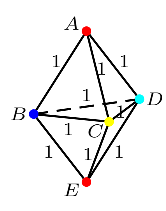
4. *Find the sum $\sum\limits_{n=1}^{\infty} \arctan \frac{1}{1+n+n^2}$.*

Solution. From the difference formula for tangent $\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$, we have that $\arctan \frac{a-b}{1+ab} = \arctan a - \arctan b$, provided that $a, b \in \mathbb{R}$ satisfy $-\pi/2 < \arctan a - \arctan b < \pi/2$. For every $n \in \mathbb{N}$ we have $\frac{1}{1+n+n^2} = \frac{1/n - 1/(n+1)}{1 + 1/(n(n+1))}$, and $0 < \arctan(1/n), \arctan(1/(n+1)) < \pi/4$, so

$$\arctan \tfrac{1}{1+n+n^2} = \arctan \tfrac{1}{n} - \arctan \tfrac{1}{n+1}.$$

Thus, $\sum\limits_{n=1}^{\infty} \arctan \frac{1}{1+n+n^2}$ is a telescopic series with terms decreasing to 0, and its sum is $\arctan 1 = \pi/4$.

5. *Each point of $\mathbb{R}^3$ is colored in one of four colors. Prove that there exist two points of the same color that are at distance 1 from each other.*



Solution. Assume that there are no two points of the same color at distance 1. Consider a hexahedron $ABCDE$ with $|AB| = |AC| = |AD| = |EA| = |EB| = |ED| = |BC| = |CD| = |BD| = 1$; then $|AE| = 2\sqrt{2/3}$. The colors of the points A, B, C, D are all different, and so are the colors of E, B, C, D; hence, A and E have the same color. This proves that any two points at the distance $r = 2\sqrt{2/3}$ of each other have the same color. Fix a point O and consider the sphere S of radius r centered at O; then all points of S have the same color as O and, clearly, there are a lot of pairs of points on S at distance 1 of each other.

6. *Find all symmetric 2025×2025 matrices all of whose entries are either 0 or 1 and such that all their eigenvalues are positive real numbers.*

Solution. Let $n = 2025$, let A be such a matrix, and let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A, listed with multiplicities. Then $\lambda_1 + \cdots + \lambda_n = \operatorname{Tr} A$ and $\lambda_1 \cdots \lambda_n = \det A$; since this product is positive and $\det A$ is an integer, we have $\det A \geq 1$. By the AGM inequality, $\frac{1}{n} \operatorname{Tr} A \geq \sqrt[n]{\det A} \geq 1$. But $\operatorname{Tr} A$, being the sum of the diagonal elements of A, is $\leq n$, so $\frac{1}{n} \operatorname{Tr} A = 1 = \sqrt[n]{\det A}$, which implies $\lambda_1 = \cdots = \lambda_n = 1$. Since A is symmetric, it is diagonalizable, so $A = I_n$.