Solutions to 2025 Rasor-Bareis examination problems

1. *Prove that* $n = 1\underbrace{00\ldots00}_{2024}1$ *is not prime.*

Solution. Using the identity $x^3+1=(x+1)(x^2-x+1)$, we have $n = 10^{2025}+1 = (10^{675})^3+1 = (10^{675}+1)\big((10^{675})^2-10^{675}+1\big)$.

Another solution. Using the identity $x^d+1=(x+1)(x^{d-1}-x^{d-2}+\cdots-x+1)$ for odd d, we have $n = 10^{2025}+1 = (10+1)\big(10^{2024}-10^{2023}+\cdots-10+1\big) = 11\cdot\underbrace{9090\cdots09}_{2023}1$

Yet another solution. We claim that n is divisible by 7. Indeed, $10^6 = 1 \bmod 7$ (by the little Fermat's theorem, or by a direct computation: $10 = 3 \bmod 7$, $10^2 = 9 = 2 \bmod 7$, $10^6 = 8 = 1 \bmod 7$). Hence, $10^{2025} = 10^{337\cdot 6+3} = 10^3 = 27 = -1 \bmod 7$, so $n = 10^{2025}+1 = 0 \bmod 7$.

2. *Evaluate* $\displaystyle\int_{-1}^{1}\frac{dx}{(e^x+1)(x^2+1)}$.

Solution. Let $f(x) = \frac{1}{(e^x+1)(x^2+1)}$, $x\in\mathbb{R}$, then $f(-x) = \frac{1}{(e^{-x}+1)(x^2+1)} = \frac{e^x}{(1+e^x)(x^2+1)}$, so $f(x)+f(-x) = \frac{1+e^x}{(e^x+1)(x^2+1)} = \frac{1}{(x^2+1)}$. Put $I=\int_{-1}^{1} f(x)\,dx$, then also $I=\int_{-1}^{1} f(-x)\,dx$, so

$$2I = \int_{-1}^{1}(f(x)+f(-x))\,dx = \int_{-1}^{1}\frac{dx}{1+x^2} = \arctan x\big|_{-1}^{1} = \pi/2,$$

and $I = \pi/4$.

3. *Prove that for every positive integer n, the number* $\sqrt[n]{\sqrt{3}+\sqrt{2}}+\sqrt[n]{\sqrt{3}-\sqrt{2}}$ *is irrational.*

Solution. We claim that for every $n\in\mathbb{N}$ there is a polynomial p_n with integer coefficients such that $x^n+x^{-n} = p_n(x+x^{-1})$. Indeed, this is true for $n=1$, and from the identity

$$x^{n+1}+x^{-(n+1)} = (x^n+x^{-n})(x+x^{-1}) - (x^{n-1}+x^{-(n-1)})$$

we have that $p_{n+1} = p_np_1 - p_{n-1}$ for all $n\geq 2$ by induction.
Let $a = \sqrt[n]{\sqrt{3}+\sqrt{2}}$; since $\sqrt{3}-\sqrt{2} = \frac{3-2}{\sqrt{3}+\sqrt{2}} = (\sqrt{3}+\sqrt{2})^{-1}$, we have $\sqrt[n]{\sqrt{3}-\sqrt{2}} = a^{-1}$. Since $p_n(a+a^{-1}) = a^n+a^{-n} = 2\sqrt{3}$ is irrational, $a+a^{-1}$ is irrational.

4. *Let $f\colon(0,\infty)\longrightarrow(0,\infty)$ be an increasing function (meaning that $x<y$ imlpies $f(x)\leq f(y)$) satisfying* $\lim\limits_{x\to\infty}\frac{f(2x)}{f(x)} = 1$. *Prove that* $\lim\limits_{x\to\infty}\frac{f(cx)}{f(x)} = 1$ *for any $c>0$.*

Solution. By induction on n, $\lim\limits_{x\to\infty}\frac{f(2^nx)}{f(x)} = 1$ for all $n\in\mathbb{N}$. Indeed, if this is true for some n, then

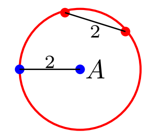
$$\lim_{x\to\infty}\frac{f(2^{n+1}x)}{f(x)} = \lim_{x\to\infty}\Big(\frac{f(2^{n+1}x)}{f(2^nx)}\cdot\frac{f(2^nx)}{f(x)}\Big) = \lim_{x\to\infty}\frac{f(2^{n+1}x)}{f(2^nx)}\cdot\lim_{x\to\infty}\frac{f(2^nx)}{f(x)} = \lim_{y\to\infty}\frac{f(2y)}{f(y)}\cdot 1 = 1.$$

Now, let $c>0$. If $c\geq 1$, let $n\in\mathbb{N}$ be such that $2^n\geq c$; then we have $f(x)\leq f(cx)\leq f(2^nx)$ for all $x>0$, $\lim\limits_{x\to\infty}\frac{f(x)}{f(x)} = \lim\limits_{x\to\infty}\frac{f(2^nx)}{f(x)} = 1$, so $\lim\limits_{x\to\infty}\frac{f(cx)}{f(x)} = 1$ by the squeeze theorem. If $0<c<1$, then with $y=cx$,

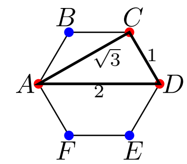
$$\lim_{x\to\infty}\frac{f(cx)}{f(x)} = \lim_{y\to\infty}\frac{f(y)}{f(c^{-1}y)} = \Big(\lim_{y\to\infty}\frac{f(c^{-1}y)}{f(y)}\Big)^{-1} = 1$$

since $c^{-1}>1$.

5. *The points of $\mathbb{R}^2$ are colored in two colors. Prove that there exists a triangle whose sides have lengths $1, \sqrt{3}, 2$ and whose vertices have the same color.*

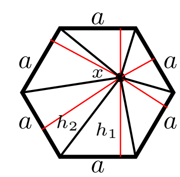


Solution. Firstly, we claim that there are two points of the same color of distance 2 of each other. Indeed, if this is not so, choose any point A, then all points on the circle S of radius 2 centered at A are colored differently from A and clearly there are two points on S of distance 2 from each other.

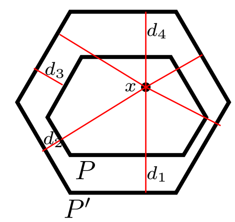


Now, assume that such a triangle doesn't exist. Choose two points A and D of the same color with $|AD| = 2$ and consider the regular hexagon $ABCDEF$. If at least one of the points B, C, E, F has the same color as A and D, then A, D and that point form the required triangle; if not, then, say, the triangle BFE has side lengths $1, \sqrt{3}, 2$.

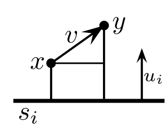
6. *Let P be an equiangular polygon (meaning that all the angles of P are equal) and let x be a point inside P. Prove that the sum of the distances from x to the lines containing the sides of P doesn't depend on the choice of x.*



Solution. The problem is easy if the polygon P is regular. In this case the triangles formed by the sides of P and the point x have base a, the distances $h_1, \ldots, h_n$ from x to the sizes of P are the heights of these triangles, so the sum $S = \frac{1}{2}a(h_1 + \cdots + h_n)$ of the areas of these triangles is the total area of P, and the sum $h_1 + \cdots + h_n = 2S/a$ doesn't depend on the choice of x.

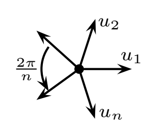


Now, the general case can be reduced to the case of a regular polygon as follows. Place P inside a regular polygon P' with the same number n of sides, the same angles $\pi(1 - 2/n)$ as P, and with sides parallel to the corresponding sides of P. For each side s_i, $i = 1, \ldots, n$, of P let d_i be the distance between s_i and the corresponding side s'_i of P'; then the distance between x and s'_i equals the distance between x and s_i plus d_i. Thus the sum of the distances between x and all sides of P is equal to the sum of the distances between x and all sides of P' minus the sum $d_1 + \cdots + d_n$, and is therefore independent of x.



Another solution. Let's introduce Cartesian coordinates on the plane. Let x and y be two points inside P, let v be the vector $y - x$. For every $i = 1, \ldots, n$ (where n is the number of sides of P) let u_i be the unit vector orthogonal to the ith side s_i of P and directed inside P; then $\mathrm{dist}(y, s_i) = \mathrm{dist}(x, s_i) + v \cdot u_i$. Hence,

$$\sum_{i=1}^{n} \mathrm{dist}(y, s_i) = \sum_{i=1}^{n} \mathrm{dist}(x, s_i) + \sum_{i=1}^{n} v \cdot u_i = \sum_{i=1}^{n} \mathrm{dist}(x, s_i) + v \cdot \sum_{i=1}^{n} u_i.$$



The vector $u = \sum_{i=1}^{n} u_i$ is invariant under the rotation of the plane by the angle of $2\pi/n$ (such a rotation simply permutes the vectors $u_1, \ldots, u_n$), hence, $u = 0$. Thus, $\sum_{i=1}^{n} \mathrm{dist}(y, s_i) = \sum_{i=1}^{n} \mathrm{dist}(x, s_i)$.