Characterizing the Sphere via Gaussian Curvature

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Abstract

A fundamental result in differential geometry states that a compact, connected, regular surface with constant Gaussian curvature K must be a sphere. In this work, we present a proof of this theorem. We show that all points on the surface are umbilical, which implies that the surface lies in a sphere and must coincide with it.

Regular Surfaces

A subset $S \subset \mathbb{R}^3$ is a *regular surface* if, for each $p \in S$, there exists a neighborhood V in $\mathbb{R}^3$ and a map $\mathbf{x}\colon U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that (Figure 1):

1. $\mathbf{x}$ is differentiable. This means that if we write
$$\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v)), \quad (u,v) \in U,$$
the functions $x(u,v)$, $y(u,v)$, $z(u,v)$ have continuous partial derivatives of all orders in U.
2. $\mathbf{x}$ is a homeomorphism. Since $\mathbf{x}$ is continuous by condition 1, this means that $\mathbf{x}$ has an inverse $\mathbf{x}^{-1}\colon V \cap S \to U$ which is continuous.
3. (The regularity condition.) For each $q \in U$, the differential $d\mathbf{x}_q\colon \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

We say that a regular surface S is *compact* if it is closed and bounded. We say S is *connected* if any two points of S can be joined by a path.

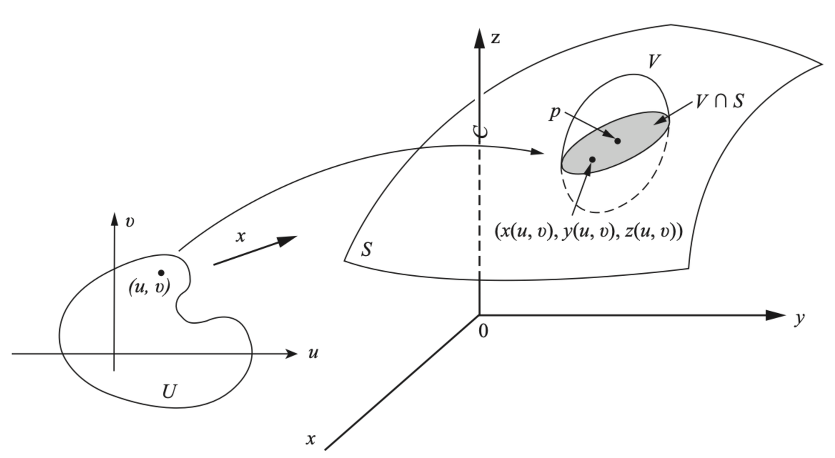

Figure 1. Regular surface

Gaussian Curvature

Let S be a regular surface. Denote by k_1 and k_2 the *principal curvatures* (i.e., the maximum and minimum normal curvature) of S. The *Gaussian curvature* is defined to be the multiplication of principal curvatures $K = k_1 k_2$.

Examples.

1. A sphere of radius r has constant Gaussian curvature $\frac{1}{r^2}$.
2. The Gaussian curvature at a saddle point is negative.

Question. Let S be a regular surface of constant curvature. When is S a sphere?

A plane is a regular surface of constant Gaussian curvature as a counterexample. While all spheres are connected, which is not true for general regular surfaces. To avoid these, we add the conditions of compactness and connectedness.

Theorem 1. Let S be a compact, connected, regular surface with constant Gaussian curvature K. Then S is a sphere.

Umbilical Points

A point $p \in S$ is called an *umbilical point* if $k_1(p) = k_2(p)$. The following proposition gives a partial solution when all points are umbilical.

Proposition 1. If all points of a connected surface S are umbilical points with positive Gaussian curvature, then S is contained in a sphere.

Proof. Let $p \in S$ and let $\mathbf{x}(u,v)$ be a parametrization of a connected coordinate neighborhood V at p. Since each $q \in V$ is an umbilical point, for any tangent vector $w = a_1\mathbf{x}_u + a_2\mathbf{x}_v$ in $T_q(S)$, $dN(w) = \lambda(q)w$ for some differentiable function $\lambda(q)$ in V. We first show that $\lambda(q)$ is constant in V. Write the above equation as
$$N_u a_1 + N_v a_2 = \lambda(\mathbf{x}_u a_1 + \mathbf{x}_v a_2).$$
Since w is arbitrary, $N_u = \lambda \mathbf{x}_u$ and $N_v = \lambda \mathbf{x}_v$. Differentiating the first equation in v and the second one in u and subtracting the resulting equations, we obtain
$$\lambda_u \mathbf{x}_v - \lambda_v \mathbf{x}_u = 0.$$
Since $\mathbf{x}_u$ and $\mathbf{x}_v$ are linear independent, we conclude that $\lambda_u = \lambda_v = 0$ for all $q \in V$. Since V is connected, λ is constant in V.

Since the Gaussian curvature $K = \det dN_p$ is positive, $\lambda \neq 0$. The point $\mathbf{x}(u,v) - (1/\lambda)N(u,v) = \mathbf{y}(u,v)$ is then fixed, because
$$\left(\mathbf{x}(u,v) - \frac{1}{\lambda}N(u,v)\right)_u = \left(\mathbf{x}(u,v) - \frac{1}{\lambda}N(u,v)\right)_v = 0.$$
Since
$$|\mathbf{x}(u,v) - \mathbf{y}|^2 = \frac{1}{\lambda^2},$$
all points of V are contained in a sphere of center $\mathbf{y}$ and radius $1/|\lambda|$. This proves the proposition locally.

To complete the proof, we observe that, since S is connected, any two points are joined by a path $\alpha : [0,1] \to S$. We can then cover the curve by coordinate charts that are contained in spheres. If the points of one of these neighborhoods are on a sphere, all the others will be on the same sphere. Since endpoints are arbitrary, all the points of S belong to this sphere. **Q.E.D.**

Elliptic Points on Compact Surfaces

Lemma 1. A regular compact surface $S \subset \mathbb{R}^3$ has at least one elliptic point.

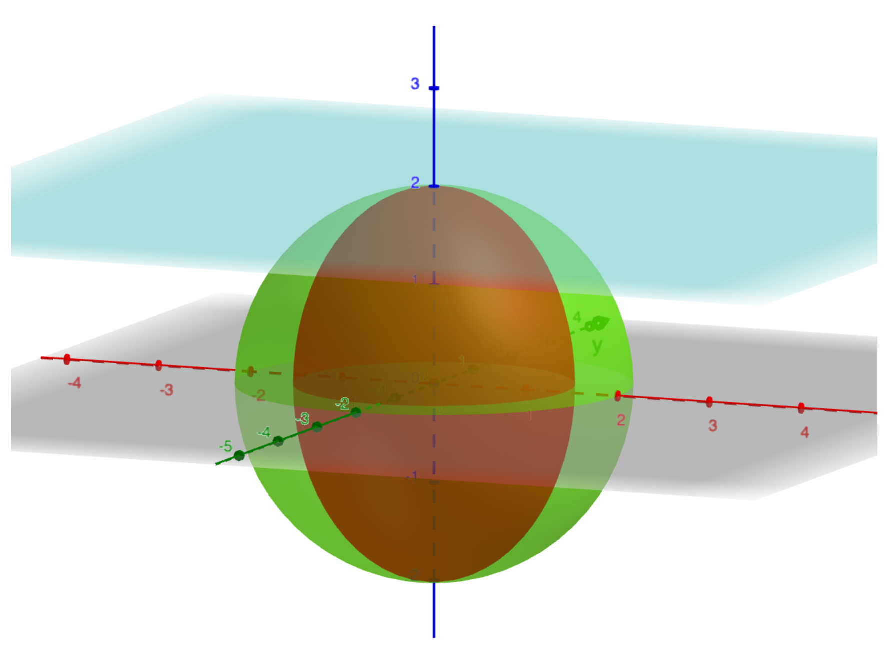

Figure 2. Σ and S are tangent at an elliptic point

Proof. Since S is compact, S is bounded. Therefore, there are spheres of $\mathbb{R}^3$, centered in a fixed point $O \in \mathbb{R}^3$, such that S is contained in the interior of the region bounded by any of them. Consider the set of all such spheres. Let r be the infimum of their radii and let $\Sigma \subset \mathbb{R}^3$ be a sphere of radius r centered in O. It is clear that Σ and S have at least one common point, say p. The tangent plane to Σ at p has only the common point p with S, in a neighborhood of p. Therefore, Σ and S are tangent at p as shown in Figure 2. By observing the normal sections at p, it is easy to conclude that any normal curvature of S at p is greater than or equal to the corresponding curvature of Σ at p. Therefore, $K_S(p) \geq K_\Sigma(p) > 0$, and p is an elliptic point, as we wished. **Q.E.D.**

Proof of Theorem

Lemma 2. Let S be a regular surface and $p \in S$ a point of S satisfying the following conditions:

1. $K(p) > 0$; that is, the Gaussian curvature in p is positive.
2. p is simultaneously a point of local maximum for the function k_1 and a point of local minimum for the function k_2 ($k_1 \geq k_2$).

Then p is an umbilical point of S.

Proof of Theorem 1. Since S is compact, there is an elliptic point, by Lemma 1. Because K is constant, $K > 0$ for all points $p \in S$.

By compactness, the continuous function k_1 on S reaches a maximum at a point $p \in S$. Since $K = k_1 k_2$ is a positive constant, k_2 is a decreasing function of k_1, and, therefore, it reaches a minimum at p. It follows from Lemma 2 that p is an umbilical point; that is, $k_1(p) = k_2(p)$.

Now, let q be any given point of S. Since we assumed $k_1(q) \geq k_2(q)$ we have that
$$k_1(p) \geq k_1(q) \geq k_2(q) \geq k_2(p) = k_1(p).$$
Therefore, $k_1(q) = k_2(q)$ for every $q \in S$.

It follows that all the points of S are umbilical and, by Proposition 1, S is contained in a sphere or a plane. Since $K > 0$, S is contained in a sphere Σ. By compactness, S is closed in Σ, and since S is a regular surface, S is open in Σ. Since Σ is connected and S is open and closed in Σ, $S = \Sigma$. Therefore, the surface S is a sphere. **Q.E.D.**

Analogue Results

Theorem 1a. Let S be a regular, compact, and connected surface with Gaussian curvature $K > 0$ and mean curvature H constant. Then S is a sphere.

The proof is entirely analogous to that of Theorem 1. Actually, the argument applies whenever $k_2 = f(k_1)$, where f is a decreasing function of k_1. More precisely, we have

Theorem 1b. Let S be a regular, compact, and connected surface of positive Gaussian curvature. If there exists a relation $k_2 = f(k_1)$ in S, where f is a decreasing function of k_1, $k_1 \geq k_2$, then S is a sphere.

Reference

[1] M. P. do Carmo, *Differential Geometry of Curves and Surfaces: Revised and Updated Second Edition*, Dover Books on Mathematics, Dover Publications, 2016.

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