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Variations on Pascal's Triangle

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Introduction

In the n-th row of Pascal's triangle, the k-th entry in each row is denoted as $\binom{n}{k}$. Define $\binom{0}{0} = 1$, $\binom{n}{n} = \binom{n}{0} = 1$ where $n \in \mathbb{N}$. The other entries in the Pascal's Triangle are defined by:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, 0 < k < n, k \in \mathbb{Z}^+.$$

By changing the last entry in each row except for the first row, we get many similar triangles. For example, changing the last entry of each row except for the first row to 2 :

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&2&&& \\ &&1&&3&&2&& \\ &1&&4&&5&&2& \\ 1&&5&&9&&7&&2 \\ &&&&\vdots&&&& \end{array}$$

Research Questions

Definition 1. Let $x \in \mathbb{N}$. Define $\binom{0}{0}_x = 1$. For $n \in \mathbb{N}$, $k = n$, define $\binom{n}{k}_x = x$. $\binom{n}{k}_x = \binom{n-1}{k}_x + \binom{n-1}{k-1}_x, 0 < k < n$. The triangle generated by $\binom{n}{k}_x$ is called the $(1, x)$-triangle.

- Is there a closed formula for the entries?
- How many times does an entry (other than 1 and x) appear in the $(1, x)$-triangle?
- How does Pascal's Triangle relate to the $(1, x)$-triangle?

How many repeats?

We start by looking for repeated entries in Pascal's triangle.

Singmaster's Conjecture: There is a finite upper bound for the maximum number of times an entry (other than one) can appear in Pascal's triangle.

We know that infinitely many entries that appear exactly six times. For example,

$$120 = \binom{120}{1} = \binom{120}{119} = \binom{16}{2} = \binom{16}{14} = \binom{10}{3} = \binom{10}{7}.$$

Also, we know that the only number known to appear eight times is 3003, which has indices:

$$\binom{3003}{1}, \binom{3003}{3002}, \binom{78}{2}, \binom{78}{76}, \binom{15}{5}, \binom{15}{10}, \binom{14}{6}, \binom{14}{8}.$$

Results

Theorem 1. For $x \in \mathbb{N}, k \in \mathbb{N}$,

$$\binom{n}{k}_x = \left(1 + \frac{(x-1)k}{n}\right) \frac{n!}{k!(n-k)!}$$

$$\binom{n}{k}_x = \left(1 + \frac{(x-1)k}{n}\right) \binom{n}{k}_1.$$

Theorem 2. Infinitely many numbers repeat exactly 3 times in the $(1, x)$-triangle.

Conjecture 1. There are entries that repeat exactly 4 times in $(1, x)$-triangle for all $x \in \mathbb{N}$.

Numerical results

The following is the table of numbers that repeat four times in $(1, x)$-Pascal's Triangle, $2 \leq x \leq 9$.

(1,x)-triangle	**maximum repeated times**	**numbers**	**check n until**
(1,2)	4	77,275,665,819,2079,2277	4000
(1,3)	4	8125	4000
(1,4)	4	1653	4000
(1,5)	4	81,196	1000
(1,6)	4	481	1000
(1,7)	4	1632	2000
(1,8)	4	297	1000
(1,9)	4	100	2000

Also, for each $(1, x)$-triangle, if we mod x for each entry, we get a Sierpinski-like triangle. Scan the QR code on the bottom left corner to see the triangle for $(1, 3)$-triangle.

References

[1] Hugo Jenkins. *Repeated Binomial Coefficients and High-degree Curves*. arXiv: 1411.4111v1 [math.NT] 2014.

[2] Singmaster, D. *"Research Problems: How often does an integer occur as a binomial coefficient?"*. American Mathematical Monthly, 78 (4): 385–386

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