Calculus X: Course Description

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Introduction

Welcome to Calculus X!
We aim to provide an accessible overview of many mathematical topics.
As an immediate result, we will examine the recovery of calculus on a manifold X.

Prerequisites

MATH 4580: Abstract Algebra

A group $(A, *)$ is a set, A, with an operation, $*$ satisfying associativity, inverses, and unitality.

An abelian group is a group where the operation is commutative.

- Homomorphism: A structure-preserving map
- Kernel: The obstruction to injectivity
- Image: A measure of surjectivity
- Isomorphism: A homomorphism that is both injective and surjective

$$\begin{tikzcd}
G \arrow[rr, "\varphi"] \arrow[d] \arrow[drr, "\varphi"] & & H \\
G/ker(\varphi) \arrow[rr, dashed, "\cong"'] & & Im(\varphi) \arrow[u]
\end{tikzcd}$$

MATH 2568: Linear Algebra

A (real) vector space $(V, +, \triangleright)$ consists of:

- $(+)$ an abelian group of vectors $(V, +)$ together with
- $(\triangleright)$ an action of scaling by the real numbers $\triangleright : \mathbb{R} \times V \to V$.

The notions of homomorphism, kernels, images, and isomorphisms all have their analogues for vector spaces.

MATH 5801: Topology

Big Idea: We measure "connectedness" in a space using open sets.

Definition: A topological space is a set S together with a topology τ satisfying the following axioms:

- (i) $S, \phi \in \tau$
- (ii) $A, B \in \tau, A \cap B \in \tau$
- (iii) $\{A_i\}_{i \in I} \in \tau, \bigcup_{i \in I} A_i \in \tau$

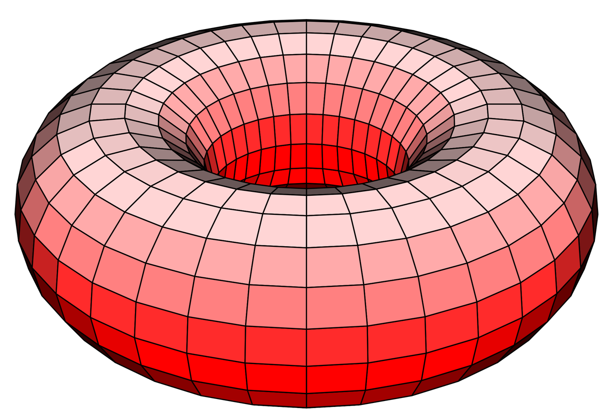
Separation $(T_0, T_1, T_2, \ldots)$ and countability axioms (first, second) are used to further describe a topological space.

Key Concepts

Manifolds

An (n-dimensional) manifold X is a topological space that locally resembles Euclidean space ($\mathbb{R}^n$). In particular, X must be:

- (ML) Locally Euclidean: $X = \bigcup U_i$ with each $U_i \cong \mathbb{R}^n$.
- (MS) Separability Axiom : Hausdorff (T_2)
- (MC) Countability Axiom: Second countable (C_2)

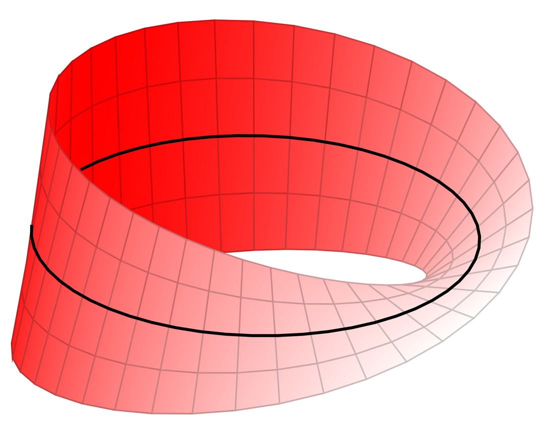


Bundles

A (vector) bundle $E \xrightarrow{\pi} B$ with fiber F consists of:

- (F) fiber (vector) space
- (B) base (topological) space
- (E) total (topological) space

so that E locally resembles the product space $F \times B$.



The space of sections $\Gamma(E)$ for a bundle $E \xrightarrow{\pi} B$ is the vector space

$$\Gamma(E) := \{s \colon B \to E \mid B \xrightarrow{s} E \xrightarrow{\pi} B = \mathrm{id}_B\}.$$

Bundle E over X	Space of sections $\Gamma(E)$
Tangent bundle TX	Vector fields $\Gamma(TX)$
Trivial line bundle $X \times \mathbb{R}$	Real-valued functions = Differential 0-forms $C^\infty(X) = \Omega^0(X)$
Cotangent bundle T^*X	Covector fields = Differential 1-forms $\Omega^1(X)$
k-wedged cotangent bundle $\bigwedge^k T^*X$	Differential k-forms $\Omega^k(X)$

Course Content

The exterior derivative d increases the degree of a differential form:

$$0 \longrightarrow \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(X) \longrightarrow 0$$

Conversely, there are free vector spaces $\Sigma_k(M) := \mathrm{span}\{\Delta^k \to M\}$ spanned by the k-simplices in M. Taking the boundary ∂ of a simplex in M decreases its dimension:

$$0 \longleftarrow \Sigma_0(X) \xleftarrow[\partial]{} \Sigma_1(X) \xleftarrow[\partial]{} \Sigma_2(X) \xleftarrow[\partial]{} \cdots \xleftarrow[\partial]{} \Sigma_n(X) \longleftarrow 0$$

We "flip" this decreasing sequence by considering the dual spaces $\Sigma^k := \Sigma_k^*$.

de Rham's Theorem

Integration can be viewed as a map $\int : \Omega^\bullet \to \Sigma^\bullet$, that is:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^0(X) \arrow[r, "d"] \arrow[d, "\int"'] & \Omega^1(X) \arrow[r, "d"] \arrow[d, "\int"'] & \Omega^2(X) \arrow[r, "d"] \arrow[d, "\int"'] & \Omega^3(X) \arrow[r, "d"] \arrow[d, "\int"'] & \cdots \\
0 \arrow[r] & \Sigma^0(X) \arrow[r, "\partial^*"'] & \Sigma^1(X) \arrow[r, "\partial^*"'] & \Sigma^2(X) \arrow[r, "\partial^*"'] & \Sigma^3(X) \arrow[r, "\partial^*"'] & \cdots
\end{tikzcd}$$

which induces an isomorphism of cohomology theories:

$$H^\bullet_{\text{deRham}}(X) \cong H^\bullet_{\text{singular}}(X).$$

Stokes' Theorem

For a differential $(n-1)$-form $\omega \in \Omega^{n-1}(X)$ on an n-dimensional manifold X with boundary ∂X,

$$\int_X d\omega = \int_{\partial X} \omega.$$

Recovering Calculus III

For a 3-dimensional oriented manifold X, there is an equivalence

$$\begin{tikzcd}
0 \arrow[r] & \Omega^0(X) \arrow[r, "d"] \arrow[d, "="'] & \Omega^1(X) \arrow[r, "d"] \arrow[d, "="'] & \Omega^2(X) \arrow[r, "d"] \arrow[d, "*"'] & \Omega^3(X) \arrow[r] \arrow[d, "*"'] & 0 \\
0 \arrow[r] & C^\infty(X) \arrow[r, "\text{grad}"'] & \Gamma(TX) \arrow[r, "\text{curl}"'] & \Gamma(TX) \arrow[r, "\text{div}"'] & C^\infty(X) \arrow[r] & 0
\end{tikzcd}$$

implying curl grad $= 0$ and div curl $= 0$.

References

Chen, Evan. *An Infinitely Large Napkin*. v1.6.20250312. Evan Chen, 2025.

Morita, Shigeyuki. *Geometry of Differential Forms*. Vol. 201, American Mathematical Society, 2010.