A Relative Variational Principle for Expanding Iterated Function Systems

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Introduction

Thermodynamic formalism provides tools to study the ergodic theory of chaotic dynamical systems. In particular, we wish to pick a distinguished measure that sees all of the chaotic behavior; i.e. a *measure of maximal entropy* or more generally an *equilibrium state*. In [1], Dr. Hemenway studied the structure of ESs in nonstationary dynamics.

During this project, we investigate a relative variational principle for expanding iterated function systems (IFSs), relating topological entropy and measure-theoretic (metric) entropy in the skew-product setting.

Stationary Case

Let X be a compact metric space and $T\colon X \to X$ be continuous. The **variational principle** states

$$h(T) := h_{\text{top}}(T) = \sup_{\nu \in M(X,T)} h_\nu(T).$$

A measure μ achieving this supremum is called a *measure of maximal entropy (MME)*.

Relative Variational Principle

Let $S\colon Y \to Y$ be continuous on a compact metric space. Suppose there is a continuous surjective map $\pi\colon X \to Y$ such that $\pi T = S\pi$. Let $\nu \in \mathcal{M}(Y,S)$. Then

$$\sup\{h_\mu(T|S) \,|\, \mu \in \mathcal{M}(X,T),\ \mu\circ\pi^{-1} = \nu\} = \int_Y h(T,\pi^{-1}y)\, d\nu(y)$$

where $h(T,\pi^{-1}y)$ denotes the relative topological entropy for T on $\pi^{-1}y$.

Theorem A

We prove the following relative variational principle for an expanding iterated function system.

Theorem 0.2. *Let (X,Φ) be an IFS of expanding type.*

$$h(\Phi) := h_{top}(\Phi) = \sup_{\nu \in \mathcal{M}(X,\Phi)} h_\nu(\Phi).$$

Iterated Function Systems

Let X be a compact metric space. Consider a collection of maps $\Phi = \{T_0, \ldots, T_{\ell-1}\}$ for which each $T_j\colon X \to X$ is a surjective map. We assume that each of these maps T_j is **expanding** in the sense that there exist $\delta > 0$ and $\gamma_j > 1$ such that whenever $0 < d(x,y) < \delta$, we have

$$d(T_j(x), T_j(y)) \geq \gamma_j \cdot d(x,y).$$

We can define **shift space** $\Sigma = \{0,\ldots,\ell-1\}^{\mathbb{N}}$ in which every element $\underline{\omega} = \omega_0\omega_1\ldots\omega_{n-1}\ldots$ is a sequence composed of letters $\mathcal{L} = \{0,\ldots,\ell-1\}$. We equip Σ with the left-shift map $\sigma\colon \Sigma \to \Sigma$ for which $\sigma(\underline{\omega}) = \omega_1\omega_2\ldots\omega_{n-1}\ldots$. We assume that (Σ,σ) has the **specification property** so that it has a unique MME (see [2]).

An **iterated function system (IFS)** is a skew product $T\colon \Sigma \times X \to \Sigma \times X$ defined

$$T(\underline{\omega}, x) = (\sigma(\underline{\omega}), T_{\omega_0}(x)).$$

For every $\underline{\omega} \in \Sigma$, define a orbit segment of length n starting at $x \in X$ by $T^n_{\underline{\omega}}(x) = T_{\omega_{n-1}} \circ \ldots \circ T_{\omega_0}(x)$.

Topological Entropy

We equip $\Sigma \times X$ with the L^1 metric; i.e.

$$d\big((\underline{\omega}, x), (\underline{\omega}', x')\big) = d_\Sigma(\underline{\omega}, \underline{\omega}') + d_X(x, x').$$

Fix $\underline{\omega} \in \Sigma$. We define the fiber above $\underline{\omega}$ by $X_{\underline{\omega}} = \{\underline{\omega}\} \times X$. For $y, x \in X_{\underline{\omega}}$, define the **Bowen distance** on X as

$$d_n(y,x) = \max_{0 \leq k \leq n} d(T^k_{\underline{\omega}} y, T^k_{\underline{\omega}} x).$$

We say a subset $E \subset X_{\underline{\omega}}$ is (ω, n, ϵ)**-separated** if for any two distinct points $x, y \in E$, $d_n(x,y) > \epsilon$. Let $s_n(\omega,\epsilon)$ denote the maximal cardinality of (ω,n,ϵ)-separated sets.

The **topological entropy of** (X,Φ) is defined as

$$h_{\text{top}}(\Phi) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log S_n(\Phi,\epsilon)$$

where

$$S_n(\Phi,\epsilon) := \frac{1}{\#\mathcal{L}_n} \sum_{\omega \in \mathcal{L}_n} s_n(\omega,\epsilon).$$

An IFS is called **chaotic** if $h_{\text{top}}(\Phi) > 0$.

Metric Entropy

Let μ be a **probability measure** on X that is **invariant** under each $T_j \in \Phi$.

Let $\xi = \{A_1, \ldots, A_k\}$ be a finite partition of X. The **entropy of** ξ is given by

$$H_\mu(\xi) = -\sum_{i=1}^{k} \mu(A_i) \log \mu(A_i).$$

The **relative entropy of** Φ with respect to ξ is defined as

$$h_\mu(\Phi,\xi) = \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{\omega \in \mathcal{L}_n} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}_{\underline{\omega}} \xi\right).$$

The **relative metric entropy** of the IFS Φ is then

$$h_\mu(\Phi) = \sup_{\xi \in \mathcal{E}} h_\mu(\Phi,\xi)$$

where $\mathcal{E}$ denotes the set of all finite partitions of X.

Ideas of Proof

We note that Ju–Lui–Yang [3] prove

$$h(\Phi) \geq \sup_{\nu \in \mathcal{M}(X,\Phi)} h_\nu(\Phi).$$

To propose an equivalence, we prove Theorem A by showing the reverse inequality.

Let $E_n \subset X_{\underline{\omega}}$ be an (ω,n,ϵ)-separated set. Define a sequence of delta measures $\nu_n = \frac{1}{|E_n|} \sum_{x \in E_n} \delta_x$ on $X_{\underline{\omega}}$ and

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} f^i_* \nu_n.$$

Pick any weak* accumulation point μ of $\{\mu_n\}_{n\in\mathbb{N}}$.

An expanding IFS (X,Φ) is **fiberwise exact**, which implies $\forall x \in X_\omega,\ \forall \epsilon > 0,\ \exists N \in \mathbb{N} \ni n \geq N$,

$$T^n_{\underline{\omega}}(B(x,\epsilon)) = T_{\omega_{n-1}} \circ \ldots \circ T_{\omega_0}(B(x,\epsilon)) = X_{\sigma^n \underline{\omega}}.$$

Given a finite partition ξ of X into balls of radius ϵ, we expect exactness to allow for a decomposition of $\xi^n = \bigvee_{k=0}^{n-1} T^{-k}_{\underline{\omega}} \xi$ for which

$$\log S_n(\Phi,\epsilon) \leq \sum_{\omega \in \mathcal{L}_n} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}_{\underline{\omega}} \xi\right)$$

which would conclude the other half of relative variational principle.

References

References

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[3] Yujun Ju, Huoxia Liu, and Qigui Yang. "Entropy of Nonautonomous Iterated Function Systems". In: *Results in Mathematics* (2024). DOI: 10.1007/s00025-024-02233-0. URL: https://doi.org/10.1007/s00025-024-02233-0.

[4] François Ledrappier and Peter Walters. "A relativised variational principle for continuous transformations". In: *Journal of the London Mathematical Society* 2.3 (1977), pp. 568–576.