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Dimensions of Diagram Modules

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Abstract

We looked at multiple different forms of diagram modules, such as Exterior Powers, Symmetric Powers, Schur Modules. Through looking at their structures, we were able to find ways to compute their dimension. We did this through code using SageMath.

Background

Fix an n-dimensional vector space $V = \mathbb{C}^n$. The **general linear group** $\mathrm{GL}(V)$ on V is the set of invertible linear transformations $V \to V$. This forms a group under composition of linear transformations.

A **representation** of $\mathrm{GL}(V)$ is a vector space W together with a multiplication map $m : GL(V) \times W \to W$ denoted by $m(g, w) = g \cdot w$ satisfying the following properties for any $g, h \in GL(V)$, $v, w \in W$:

(1) $g(hw) = (gh)w$ (2) $g \cdot (v + w) = g \cdot v + g \cdot w$ (3) $e \cdot x = x$

Diagram Modules

A **diagram** D is a finite subset of $\mathbb{N} \times \mathbb{N}$. We think of a diagram as the sequence of its columns $D = (\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \cdots)$, where $\mathbf{a}^{(j)} = (a_1^{(j)} < \cdots < a_{r_j}^{(j)})$ is the list of rows in column j.

A **n-filling** of a diagram D is a labeling τ of D with values from $[n] = \{1, \dots, n\}$. We think of $\tau = (\mathbf{b}^{(1)}, \mathbf{b}^{(2)}, \dots)$ in terms of columns, so $\mathbf{b}^{(j)} = (b_1^{(j)}, \dots, b_{r_j}^{(j)})$ denotes the fillings in column j.

Given a filling τ, we define a polynomial

$$\Delta_{D,\tau} := \Delta_{\mathbf{a}^{(1)},\mathbf{b}^{(1)}} \cdots \Delta_{\mathbf{a}^{(r)},\mathbf{b}^{(r)}}$$

where for two sequences $\mathbf{a} = (a_1, \dots, a_r)$ and $\mathbf{b} = (b_1, \dots, b_r)$,

$$\Delta_{\mathbf{a},\mathbf{b}} := \det \begin{pmatrix} x_{a_1 b_1} & x_{a_1 b_2} & \cdots & x_{a_1 b_r} \\ x_{a_2 b_1} & x_{a_2 b_2} & \cdots & x_{a_2 b_r} \\ \vdots & \vdots & \ddots & \vdots \\ x_{a_r b_1} & x_{a_r b_2} & \cdots & x_{a_r b_r} \end{pmatrix}$$

The **diagram module** V^D of D is the subspace of the space of polynomials $\mathbb{C}[x_{ij}]$ $(1 \le i \le \infty, 1 \le j \le n)$ spanned by $\Delta_{D,\tau}$ as τ ranges among all fillings of D.

For any $A = (M_{jk})_{j,k=1}^n \in GL(V)$, the $GL(V)$ acts on the polynomial ring $\mathbb{C}[x_{ij}]$ with:

$$A \cdot x_{ij} = \sum_{j=1}^{n} M_{kj} x_{ik}$$

Question

What is the dimension of V^D for various diagrams D?

Known Cases

I. Symmetric Powers: $\mathrm{Sym}^k V = V^D$ where D is a single row with k boxes.

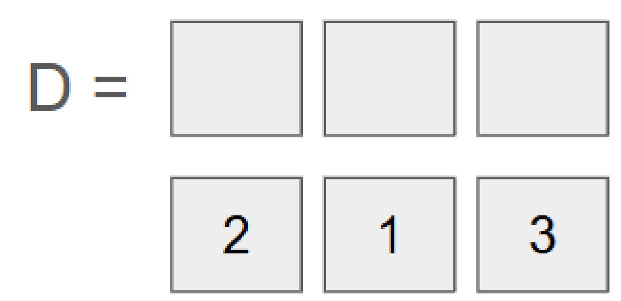


Figure 1. An example Diagram with fillings up to 3

So the polynomial corresponding to this filling of this diagram is:

$$\Delta_\tau = \Delta_{(1),(2)} \Delta_{(1),(1)} \Delta_{(1),(3)} = x_{12} x_{11} x_{13}$$

V^D is congruent to the span$\{x_{11}^{(a_1)} x_{12}^{(a_2)} \cdots x_{1n}^{(a_n)} : a_1 + \cdots + a_n = k\}$ It is known that the dimension of $\mathrm{Sym}^k V$ is $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$, so for this case V^D has a dimension of $\binom{n+2}{3}$

II. Exterior powers: $\Lambda^k V$ where D is a single column with k boxes.

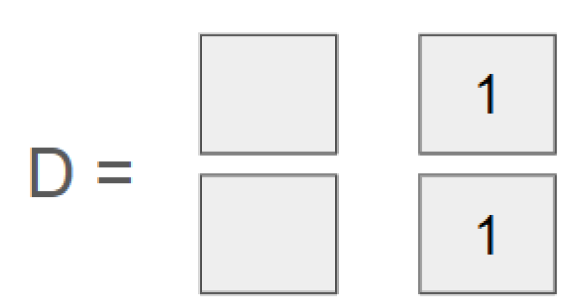


Figure 2. An example Diagram with fillings up to 2

The polynomial corresponding to the example filling is:

$$\Delta_\tau = \Delta_{(1,2),(1,1)} = \det\begin{pmatrix} x_{11} & x_{11} \\ x_{21} & x_{21} \end{pmatrix} = 0$$

$\Lambda^k V$ is congruent to the span$\{\Delta_{\underline{a}}\}$, and the exterior power is known to have a dimension of $\binom{n}{k}$. So if V is n n-dimensional vector space, the dimension of $\Lambda^2 V$ is $\binom{n}{2}$.

III. Schur modules (D is the diagram of a partition λ)

A **partition** is a decreasing sequence $\lambda = (\lambda_1 \ge \cdots \ge \lambda_m)$. When D is the diagram of a partition λ, $V^D = V^\lambda$ is **Schur module**.

A **semistandard Young tableaux** is defined as fillings that weakly increase along each row and strictly increase along each column.

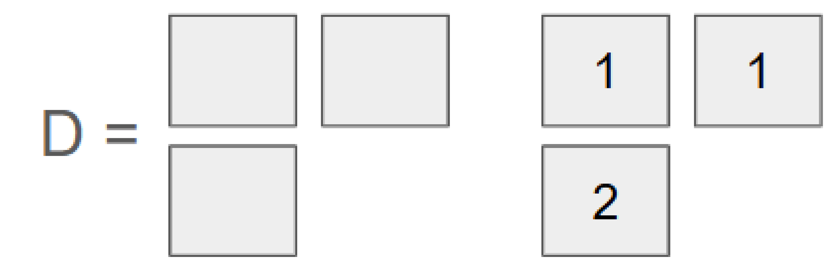
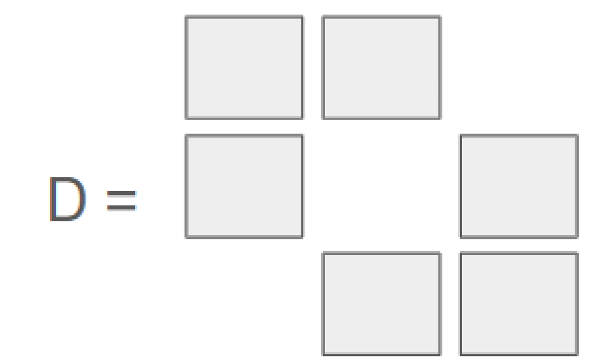


Figure 3. An example Diagram with $\lambda = (2, 1)$ and fillings up to 2

The dim V^λ is the number of semistandard Young tableaux of shape λ and fillings in $\{1, \dots, n\}$. So for the diagram above the dimension would be 2 for $n = 2$.

Results

We have written code in SageMath (Python) to compute the dimensions of diagram modules.



For example we used the code to calculate the dimensions of V^D using the diagram above.

dim V:	1	2	3	4	5
dim V^D:	0	1	27	188	785

This code can be generalized to other diagram modules and other dimensions allowing for us to generate a formula of this diagram's dimensions.

Future

In the future we could look at other forms of diagram modules and try to find formulas that describe their dimension such as **flagged diagram modules** $FM(D)$.

To get this diagram module, if D has n rows $FM(D)$ is the image of V^D under the map $x_{ij} \to 0$ for $j > i$, $x_{ij} \to x_{ij}$ for $j \le i$.

When D is $D(w)$ for a permutation $w \in S_n$, $FM(D)$ is called the **Schubert module**. In this case, the dimension is known to be the number of certain diagrams for the permutation w called pipe dreams.

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