A Diagrammatic Presentation of the Temperley-Lieb Algebra

Neal Butcher

Department of Mathematics, The Ohio State University

DEPARTMENT OF MATHEMATICS

Abstract

The Temperley-Lieb Algebra specifically emphasizes the connections of diagrammatic and algebraic presentations. In this poster, we will give motivations of defining diagrammatic Temperley-Lieb-Kauffman (TLK) algebra arising algebraically, which is the Temperley-Lieb-Jones (TLJ) algebra correspondingly. Then, we will describe the structure of TLK algebra. We will also introduce TLK planar algebra operations with some examples. Surprisingly, the diagrammatic operations (/calculations) in other mathematical subjects such as quantum topology. Hence, at the end of the poster, we will introduce the work that we are currently working with and the work that can be done in the future: the Reidemeister moves encoding in the diagrammatic planar operations, and many other applications.

Preliminary Discussions

Construction of the Diagrammatic Temperley-Lieb Algebra

We define $TLK_n(d)$ to be the complex vector space with some extra structures: Given A standard n-box is a rectangle with 2n nodes, labeled as follows:

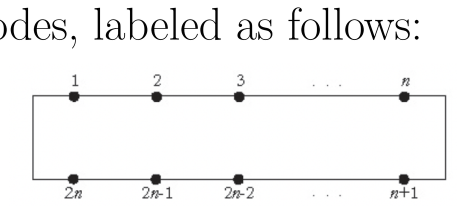


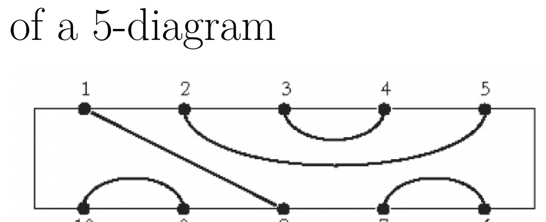
Figure: Standard n-box

An n-diagram is a graph drawn on the nodes of a standard n-box such that

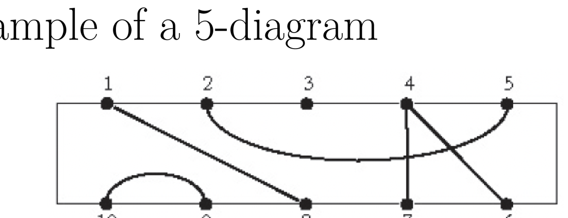
- Every node is connected to exactly one other node by a single edge.
- All edges must be drawn inside the n-box.
- The graph can be drawn so that no edges cross.

Example and Non-example of Diagrammatic the Temperley-Lieb Algebra

- Example of a 5-diagram



- Non-example of a 5-diagram



Main Results

Definitions

- **Jones' algebraic Temperley-Lieb algebras**

The Jones' algebraic TL algebra $TLJ_n(d)$ as the unital $\star$-algebra generated by $1, e_1, \ldots, e_{n-1}$ subject to the following relations:
(J_1) $e_i^2 = e_i = e_i^*$ for all $i = 1, \ldots, n-1$
(J_2) $e_i e_j = e_j e_i$ for all $|i-j| > 1$
(J_3) $e_i e_{i\pm1} e_i = d^{-2} e_i$.

- **Kauffman's diagrammatic Temperley-Lieb (TLK) algebras:**

The $TLK_n(d)$ is defined as the complex vector space whose standard basis is the set of the non-intersecting string diagrams on a rectangle with n boundary points on the top and bottom.

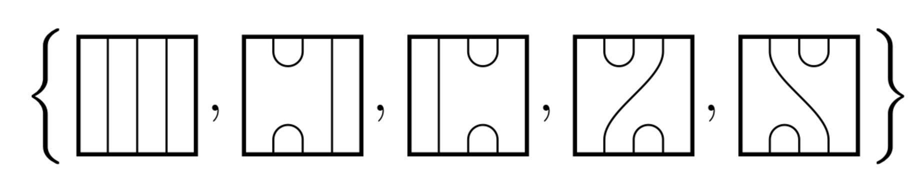


Figure: Basis for $TLK_3(d)$

On $TLk_n(d)$, multiplication is defined by the stacking of boxes on top of each other. This stacking is followed by the exclusion of strings that loop on themselves, with the multiplication of d if this occurs, and the consideration of strings as if they were on a single bracket. The resultant of this multiplication can be expressed on a single bracket.

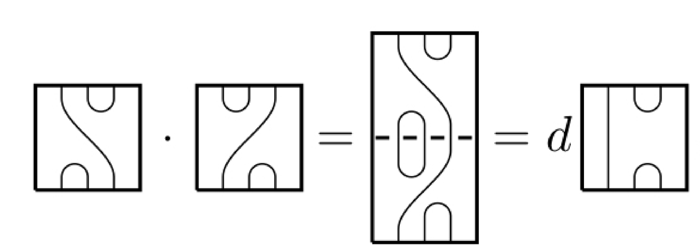
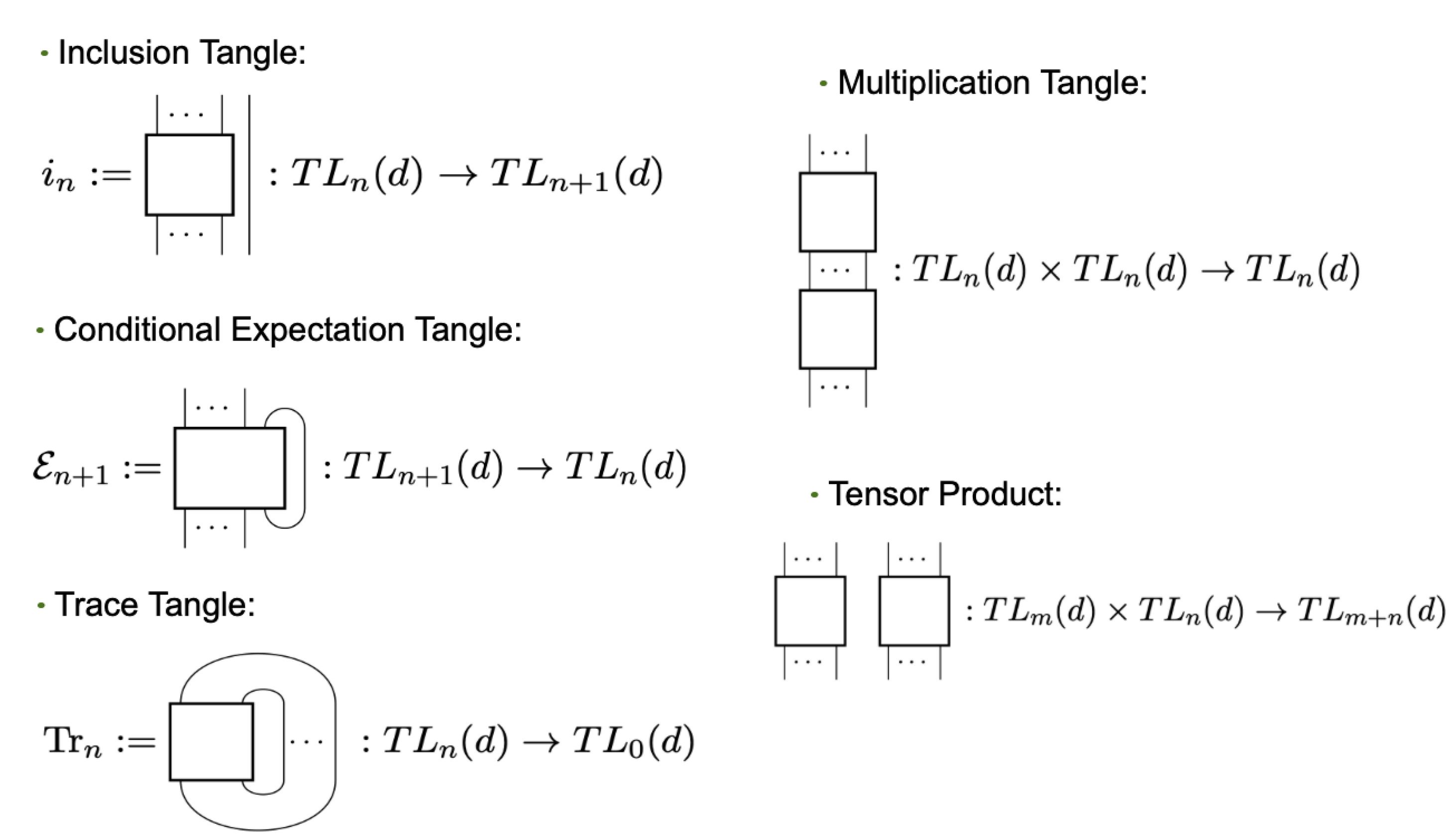


Figure: Example of the multiplication of two elements in $TLK_3(d)$

Planar Operations on the TL-Algebra



Examples Planar Operations on the TL-Algebra

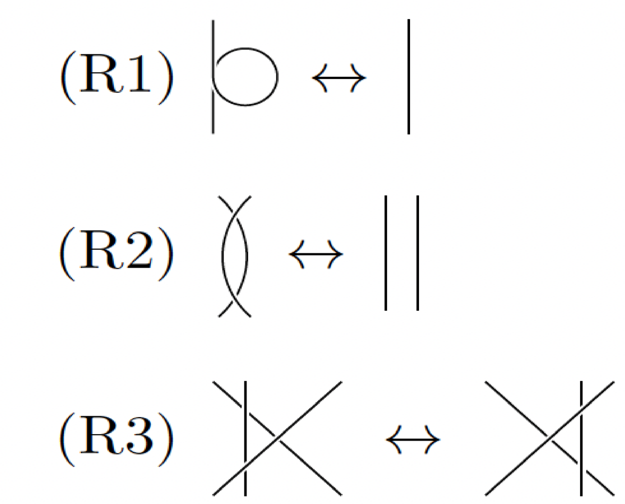
- $\mathcal{E}_{n+1} \circ i_n = d \cdot id_n$.
- $Tr_{n+1} = Tr_n \circ \mathcal{E}_{n+1}$
- $Tr_{n+1}(i_n(x) \cdot E_n) = Tr_n(x)$ for all $x \in Tr(d)$

Future Applications

The Temperley-Lieb algebras can be used to construct a polynomial invariant of knots and links. We will begin by illustrating the basis by defining the Kauffman bracket for knot projections. First, we will introduce Reidemeister's theorems.

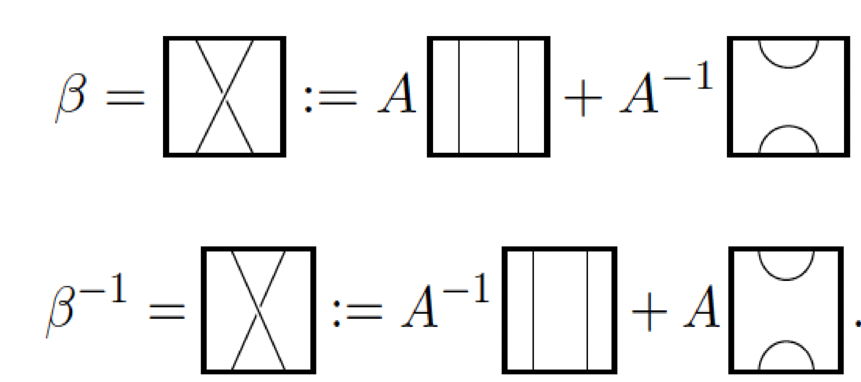
Reidemeister moves

Two knot/link projections represent isotopic knots if and only if they are related by a finite number of the Reidemeister moves:



The Kauffman bracket

- Suppose $d = -A^2 - A^{-2}$. Define the crossings $\beta^{\pm1}$ in TL_2 by:



Proving Relationships Through Diagrams:

- $\beta\beta^{-1} = \beta^{-1}\beta = id_2 \in TL_2$

References

- Exercises in quantum algebra. David Penneys, 17 August 2020.
- Diagram Calculus for the Temperley-Lieb Algebra. Dana Ernst, Graduate Student Combinatorics Conference 2007.

Acknowledgments

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