There exist t_0 such that every t_0-tough graph is Hamiltonian

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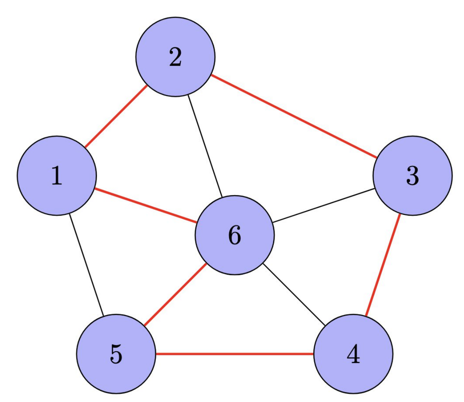
Background



Hamiltonian Cycle

A graph G is Hamiltonian if it contains a Hamilton cycle (a cycle contain every vertex of G)

If G is a Hamiltonian graph, then

$$k(G-S) \leq |S|$$

for every nonempty proper subset S of $V(G)$

So, if G is Hamiltonian:

$$\frac{|S|}{k(G-S)} \geq 1$$

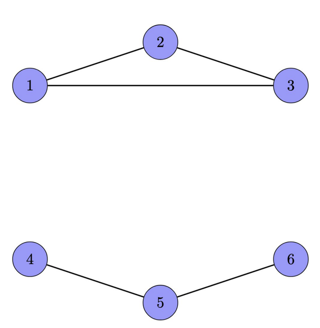
for every nonempty proper subset S of $V(G)$

Graph Toughness

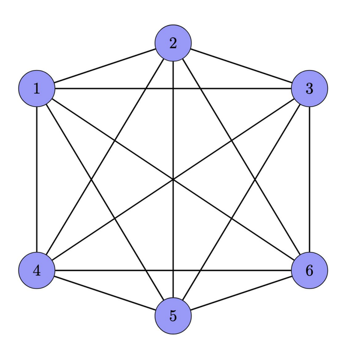
A noncomplete graph G is t-tough if

$$t(G) = \min \frac{|S|}{k(G-S)}$$

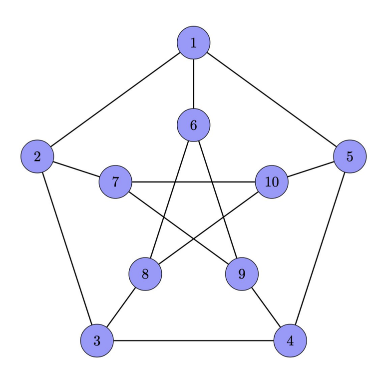
G: a graph
S: none-empty proper subset of G
t: toughness of the graph
$|S|$: the number of vertex in subset S
$k(G-S)$: number of components of subgraph $(G-S)$



A graph G is disconnected if and only if $t(G)$=0



A graph G is complete if and only if $t(G)$=+ ∞

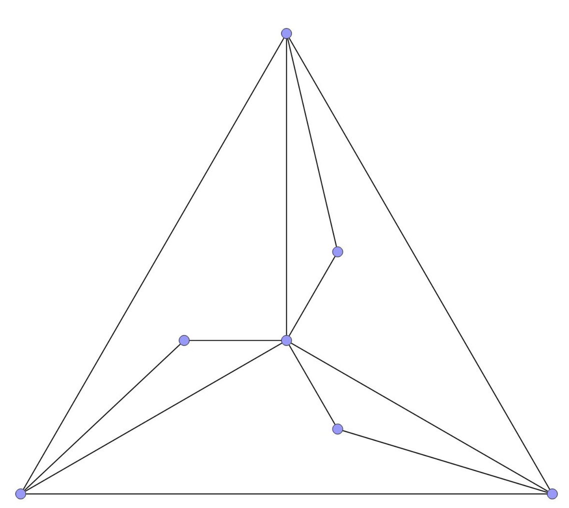


Peterson graph P is a 1-tough graph. In fact, $t(P) = \frac{4}{3}$.

V. Chvátal

Every Hamiltonian graph is 1 tough since $\frac{|S|}{k(G-S)} \geq 1$ for every nonempty proper subset S of $V(G)$.
However, noted by Chvátal, the converse is not true.

Graph H is 1 tough
Graph H is non-Hamiltonian



Conjecture: Every t-tough graph with $t > \frac{3}{2}$ is Hamiltonian

If G is not complete, then $t \leq \frac{1}{2}\kappa$

Therefore, every t-tough graph with $t >$ is 4-connected

According to Tutte's theorem, every four connect planar graphs having at least edges has a Hamiltonian circuit

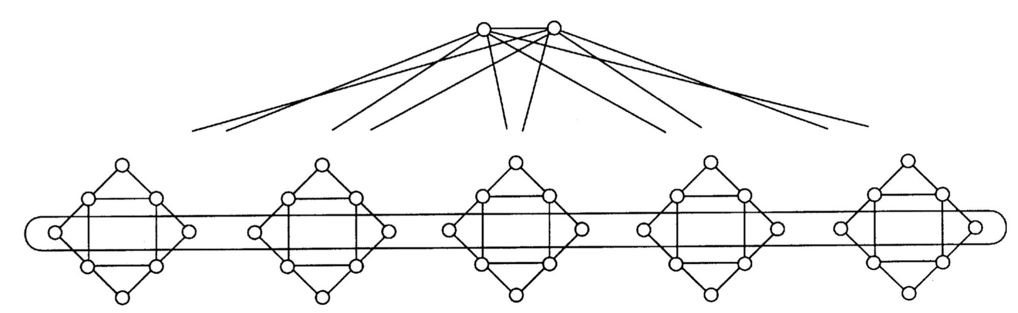
D. Bauer, H.J. Broersma, H.J. Veldman

For a given graph H and two vertices x and y of H define $G(H,x,y,l,m)$ $[l,m \in \mathbb{N}]$ as:

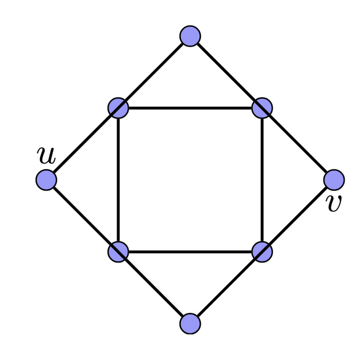
Take m disjoint copies $H_1, \cdots, H_m$ of H, with x_i, y_i the vertices in H_i corresponding to the vertices x, y in H $(i = 1, \cdots, m)$.

Let F_m be the graph obtained from $H_1 \cup \cdots \cup H_m$ by adding all possible edge between pairs of vertices in $\{x_1, \cdots, x_m, y_1, \cdots, y_m\}$.

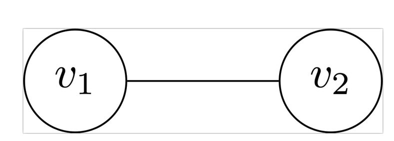
Let $T = K_l$ and let $G(H,x,y,l,m)$ be the join $T \vee F_m$ of T and F_m.



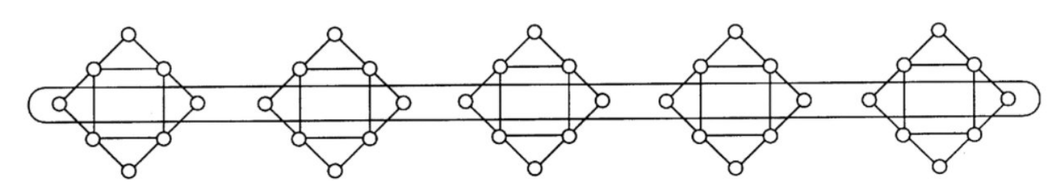
$G(L,u,v,2,5)$



Subgraph L



$T = K_2$



F_5: 5 copies of subgraph every vertex in the circle is connected to each other

Theory: Let H be a graph and x, y two vertices of H which are not connected by a Hamiltonian path of H. If $m \geq 2l + 3$ then $G(H,x,y,l,m)$ is nontraceable.

Proof by Contradiction:

Suppose $G(H,x,y,l,m)$ contains a Hamilton path P.

The intersection of P and F_m consists of a collection $\mathscr{P}$ of at most $l + 1$ disjoint paths, together containing all vertices in F_m.

Since $m \geq 2(l + 1) + 1$

There must exist a subgraph of H_{i0} in F_m such that no end vertex of a path $\mathscr{P}$ lies in H_{i0}.

Intersection of P and H_{i0} is a path with end-vertices x_{i0} and y_{i0} that contains all vertices of H_{i0}.

This contradicts the fact that H_{i0} is a copy of the graph H without a Hamilton path between x and y.

Theory: For $l \geq 2$ and $m \geq 1$

$$t(G(L,u,v,l,m)) = \frac{l + 4m}{2m + 1}$$

Proof:

Let $G = G(L,u,v,l,m)$ for some $l \geq 2$ and $m \geq 1$, and choose $S \subseteq V(G)$ such that $k(G-S) > 1$ and $t(G) = \frac{|S|}{k(G-S)}$.

$V(T) \subseteq S$.

Define $S_i = S \cap V(L_i)$, $s_i = |S_i|$ and let k_i be the number of components of $L_i - S_i$ that contains neither u_i nor v_i $(i = 1, \cdots, m)$.

$$k(G) = \frac{l + \sum_{i=1}^{m} s_i}{c + \sum_{i=1}^{m} k_i} \geq \frac{l + \sum_{i=1}^{m} s_i}{1 + \sum_{i=1}^{m} k_i}$$

where:

$$c = \begin{cases} 0, & \text{if } u_i, v_i \in S_i \text{ for all } i \in \{1, \cdots, m\} \\ 1, & \text{otherwise} \end{cases}$$

To justify the equality, we shall prove $s_i \geq 2k_i (i = 1, \cdots, m)$

Note that $k_i \leq 2$, since $L - \{u,v\}$ has independence number 2; $s_i \geq 2k_i$ if $k_i = 0$ or $k_i = 1$.By exhaustion it is readily checked that if $s_i \leq 3$, then $k_i \leq 1$ ($s_i \geq 2k_i$ if $k_i = 2$).

$$t(G) \geq \frac{l + 2\sum_{i=1}^{m} k_i}{1 + \sum_{i=1}^{m} k_i}$$

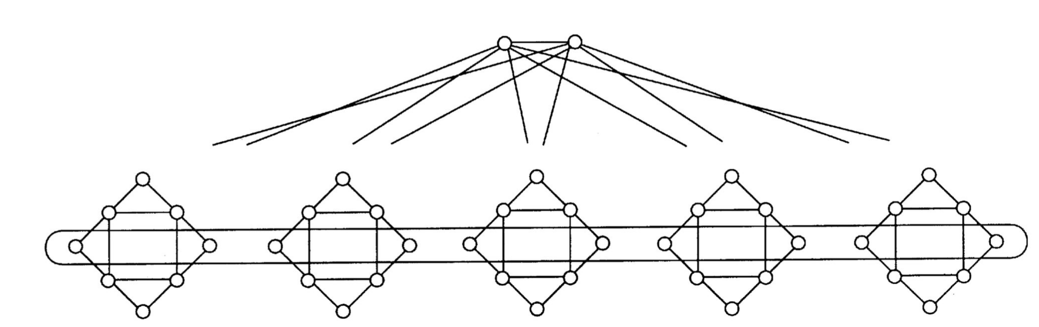
Since $l \geq 2$, the lower bound for $t(G)$ is a nonincreasing function of $\sum_{i=1}^{m} k_i$ and is minimized if $k_i = 2$ for all $i \in \{1, \cdots, m\}$

$$t(G) \geq \frac{l + 4m}{1 + 2m}$$

Set $U = V(T) \cup U_1 \cup \cdots \cup U_m$, where U_i is the set of vertices of L_i having degree 4 in L_i $(i = 1, \cdots, m)$ The proof is completed by observing that

$$t(G) \leq \frac{|U|}{k(G-U)} = \frac{l + 4m}{1 + 2m}$$

Corollary: For every $\varepsilon > 0$ there exists a $\left(\frac{9}{4} - \varepsilon\right)$-tough nontraceable graph.



$G(L,u,v,2,5)$

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