Derivation of the Nonlinear Schrödinger Equation

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The Equation

The nonlinear Schrödinger equation (NLSE) is a class of second-order nonlinear partial differential equations whose solution is a complex wave. It is given by:

$$i\frac{\partial\psi}{\partial t} + \Delta\psi \pm |\psi|^{2\sigma}\psi = 0$$

Whether the nonlinear term is positive or negative dictates whether the NLSE is focusing or defocusing respectively. This equation is used to model light propogation through optical fibers, deep-water waves, and Bose-Einstein condensates.

Derivation Part I: Maxwell's Equations

This fall semester, we chose to derive the NLSE in the context of nonlinear optics. For this, we begin with Maxwell's equations and flux density relations in a vaccuum:

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$
$$\mathbf{D} = \epsilon_0 \mathbf{E}$$
$$\mathbf{B} = \mu_0 \mathbf{H}$$

After rearranging and utilizing the following vector identity fitted to our scenario:

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \Delta \mathbf{E}$$

one can arrive at the vector wave equation, which can further be decoupled into n-scalar wave equations for an n-dimensional problem. A common simplification used is to suppose that the electric field vector is pointed solely in one direction (linearly polarized), reducing us down to one scalar wave equation:

$$\nabla^2 E = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

Given this equation, we can furthermore take this expression and look for a continuous wave beam solution of the form:

$$E(x,y,z,t) = \mathcal{E}(x,y,z)e^{-i\omega_0 t} + c.c.$$

where $c.c$ is the complex conjugate. When plugged into our wave equation, we yield the scalar Helmholtz equation, or the time-independent wave equation

$$\nabla^2\mathcal{E} + k_0^2\mathcal{E} = 0$$

where our constant is rewritten given the following dispersion relation in a vacuum

$$k_0^2 = \frac{\omega_0^2}{c^2}$$

Our next objective is to apply this to a nonlinear system.

Derivation Part II: Apply Nonlinearity

Now, we must transfer this to a system with Kerr nonlinearity (such as a weakly nonlinear optical cable). To do this, we must alter our dispersion relation to account for the Kerr effect, which alters our refractive index and therefore the wave number

$$k^2 = \frac{\omega_0^2}{c^2}n^2 = \frac{\omega_0^2}{c^2}n_0^2\left(1 + \frac{4n_2}{n_0}|\mathcal{E}|^2\right) = k_0^2\left(1 + \frac{4n_2}{n_0}|\mathcal{E}|^2\right)$$

Now, we apply this to our Helmholtz equation

$$\nabla^2\mathcal{E} + k^2\mathcal{E} = 0$$
$$\nabla^2\mathcal{E} + k_0^2\mathcal{E}\left(1 + \frac{4n_2}{n_0}|\mathcal{E}|^2\right) = 0$$
$$\nabla^2\mathcal{E} + k_0^2\mathcal{E} + k_0^2\frac{4n_2}{n_0}|\mathcal{E}|^2\mathcal{E} = 0$$

Now, we make the following substitution

$$\mathcal{E}(x,y,z) = e^{ik_0 z}\psi(x,y,z)$$

into our nonlinear Helmholtz equation. This yields

$$2ik_0\psi_z - k_0^2\psi_{zz} + \psi_{yy} + \psi_{xx} + k_0^2\frac{4n_2}{n_0}|e^{ik_0 z}\psi|^2\psi = 0$$
$$2ik_0\psi_z - k_0^2\psi_{zz} + \psi_{yy} + \psi_{xx} + k_0^2\frac{4n_2}{n_0}|\psi|^2\psi = 0$$
$$2ik_0\psi_z - k_0^2\psi_{zz} + \nabla_\perp^2\psi + k_0^2\frac{4n_2}{n_0}|\psi|^2\psi = 0$$

We further suppose that this is a para-axial plane wave, which means it propagates in a slow-varying fashion along one direction (usually taken to be the z-axis). Given this, we apply the paraxial approximation, which takes the second partial derivative of the wave with respect to the propagation direction as insignificant

$$2ik_0\psi_z + \nabla_\perp^2\psi + k_0^2\frac{4n_2}{n_0}|\psi|^2\psi = 0$$

Here, we derived a $3+0$ nonlinear Schrödinger equation.

Graph

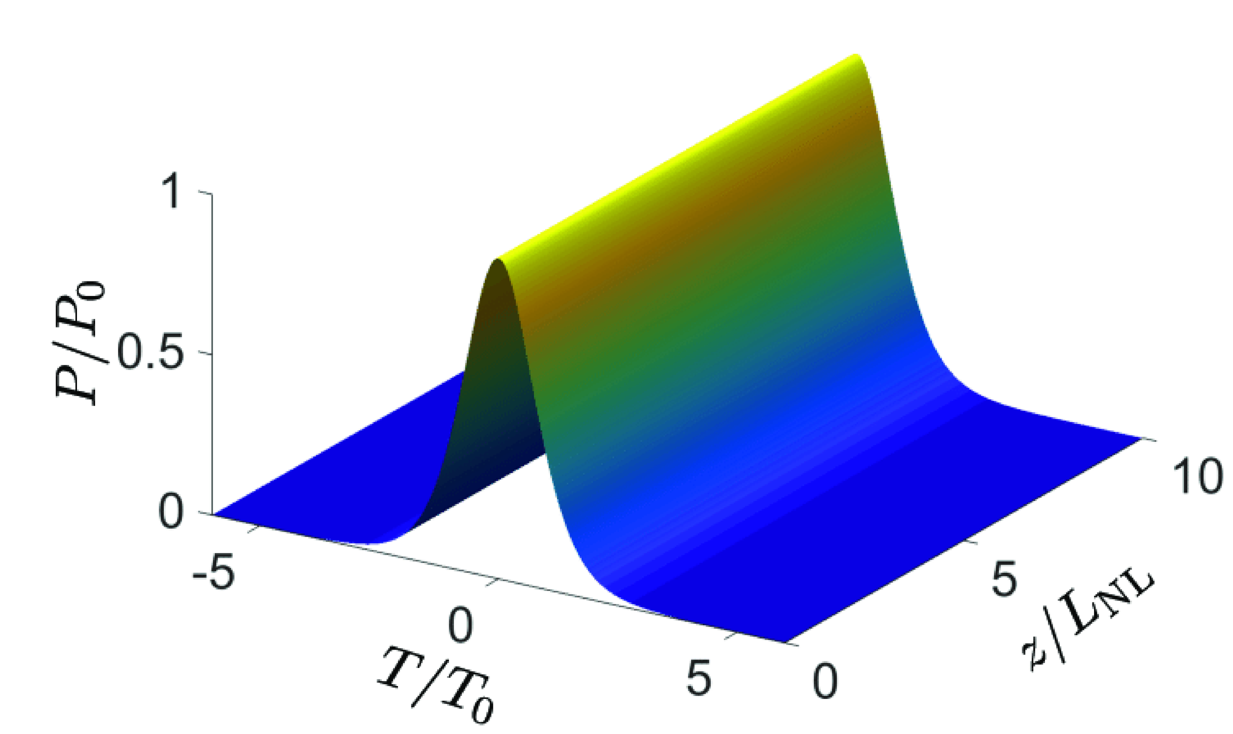

Figure 1: Graph of the fundamental soliton to the NLSE

Acknowledgment

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Conservation Law Equation and Use

The spring semester was partly dedicated to applying conservation laws in the context of partial differential equations and the NLSE. Firstly, we have the following initial value problem

$$i\psi_t + \nabla^2\psi + |\psi|^{2\sigma}\psi = 0; \psi(\mathbf{x},0) = \varphi(\mathbf{x})$$

Along with the following definitions

- Mass $= \int_{\mathbf{R}^n}|\psi|^2 d\mathbf{x}$
- Hamiltonian Density $= \mathcal{H} = \nabla\psi \cdot \nabla\overline{\psi} - \frac{|\psi|^{2\sigma+2}}{\sigma+1}$
- Hamiltonian $= H = \int_{\mathbf{R}^n}\mathcal{H}d\mathbf{x}$

Conservation laws take the following form

$$\frac{\partial}{\partial t}u(\mathbf{x},t) + \nabla\cdot\mathbf{f} = 0$$

Where the function $u(\mathbf{x},t)$ is chosen and if we can find vector $\mathbf{f}$ that satisfies this, we have a conservation law. From here, we can apply this to mass and the Hamiltonian. Worked out, the conservation law and the vector $\mathbf{f}$ for each case will be

$$\frac{\partial}{\partial t}|\psi|^2 - \nabla\cdot i(\overline{\psi}\nabla\psi - \psi\nabla\overline{\psi}) = 0$$

$$\frac{\partial}{\partial t}\mathcal{H} - \nabla\cdot i(\Delta\psi\nabla\overline{\psi} - \Delta\overline{\psi}\nabla\psi + |\psi|^{2\sigma}\psi\nabla\overline{\psi} - |\psi|^{2\sigma}\overline{\psi}\nabla\psi) = 0$$

Now we can make the following step

$$\int_{\mathbf{R}^n}\frac{\partial}{\partial t}|\psi|^2 d\mathbf{x} - \int_{\mathbf{R}^n}\nabla\cdot i(\overline{\psi}\nabla\psi - \psi\nabla\overline{\psi})d\mathbf{x} = 0$$

$$\int_{\mathbf{R}^n}\frac{\partial}{\partial t}\mathcal{H}d\mathbf{x} - \int_{\mathbf{R}^n}\nabla\cdot i(\Delta\psi\nabla\overline{\psi} - \Delta\overline{\psi}\nabla\psi + |\psi|^{2\sigma}\psi\nabla\overline{\psi} - |\psi|^{2\sigma}\overline{\psi}\nabla\psi)d\mathbf{x} = 0$$

Utilizing divergence theorem and the fact that the solution and its variations go to 0 as $\mathbf{x}$ goes to ∞, both the divergence integrals go to 0

$$\int_{\mathbf{R}^n}\frac{\partial}{\partial t}|\psi|^2 d\mathbf{x} = 0$$

$$\int_{\mathbf{R}^n}\frac{\partial}{\partial t}\mathcal{H}d\mathbf{x} = 0$$

From here, one can quickly see that this becomes

$$\frac{d}{dt}\int_{\mathbf{R}^n}|\psi|^2 d\mathbf{x} = 0$$

$$\frac{d}{dt}H = 0$$

This ensure both quantities are conserved in our NLSE system.

Key

$\mathbf{E}$ = Electric Field
$\mathbf{D}$ = Electric Field Flux Density
$\mathbf{H}$ = Magnetic Field
$\mathbf{B}$ = Magnetic Field Flux Density
c = Speed of Light
ω = Frequency
k = Wave Number
n =Index of Refraction
n_2 =Kerr Coefficient