Thresholds in Random Graphs

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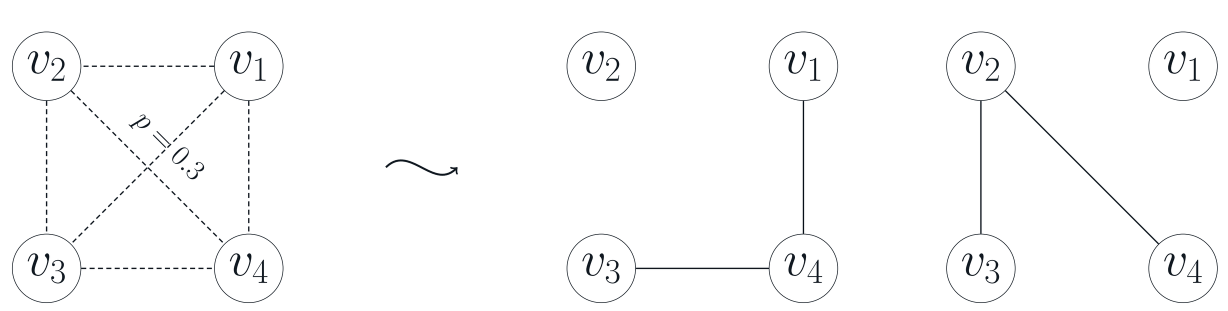
Introduction

Graphs (also known as networks) are mathematical structures with a variety of real-world applications, from modeling disease spread to analyzing portfolio diversification. In practice, some applications deal with uncertain outcomes, and so they are better modeled by a so-called *random graph*, which incorporates uncertainty into its structure.

We investigated the connectedness and planarity of random graphs by using Python to simulate different combinations of n, number of vertices, and p, the probability for an edge to form.

Definitions

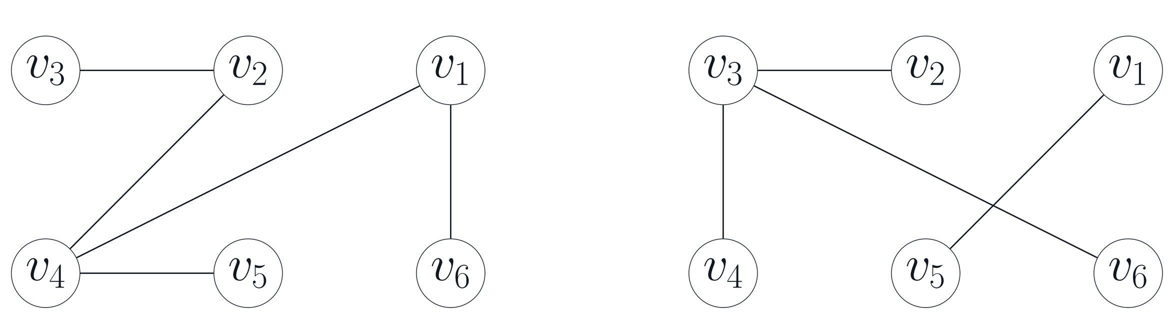
Definition 1. A *random graph* G sampled from the distribution $G(n, p)$ has n vertices and between each pair of vertices there exists an edge with probability p.



Two random graphs sampled from $G(4, 0.3)$

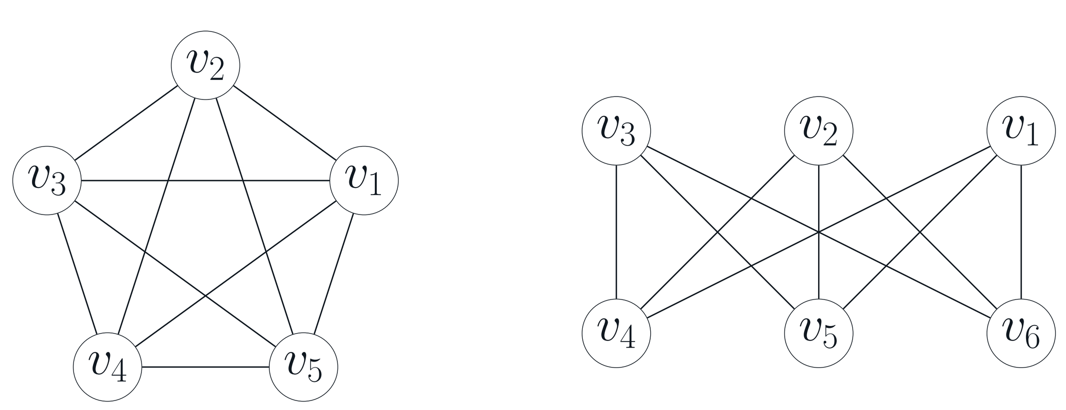
Definition 2. An *adjacency matrix* of a graph G is a matrix A where rows and columns are labeled by graph vertices. If there is an edge between v_i and v_j in G, the entry a_{ij} in A is 1; the entry is 0 if there is no edge.

Definition 3. A graph G is *connected* if any vertex of G can be reached from any other vertex of G by a path of edges.



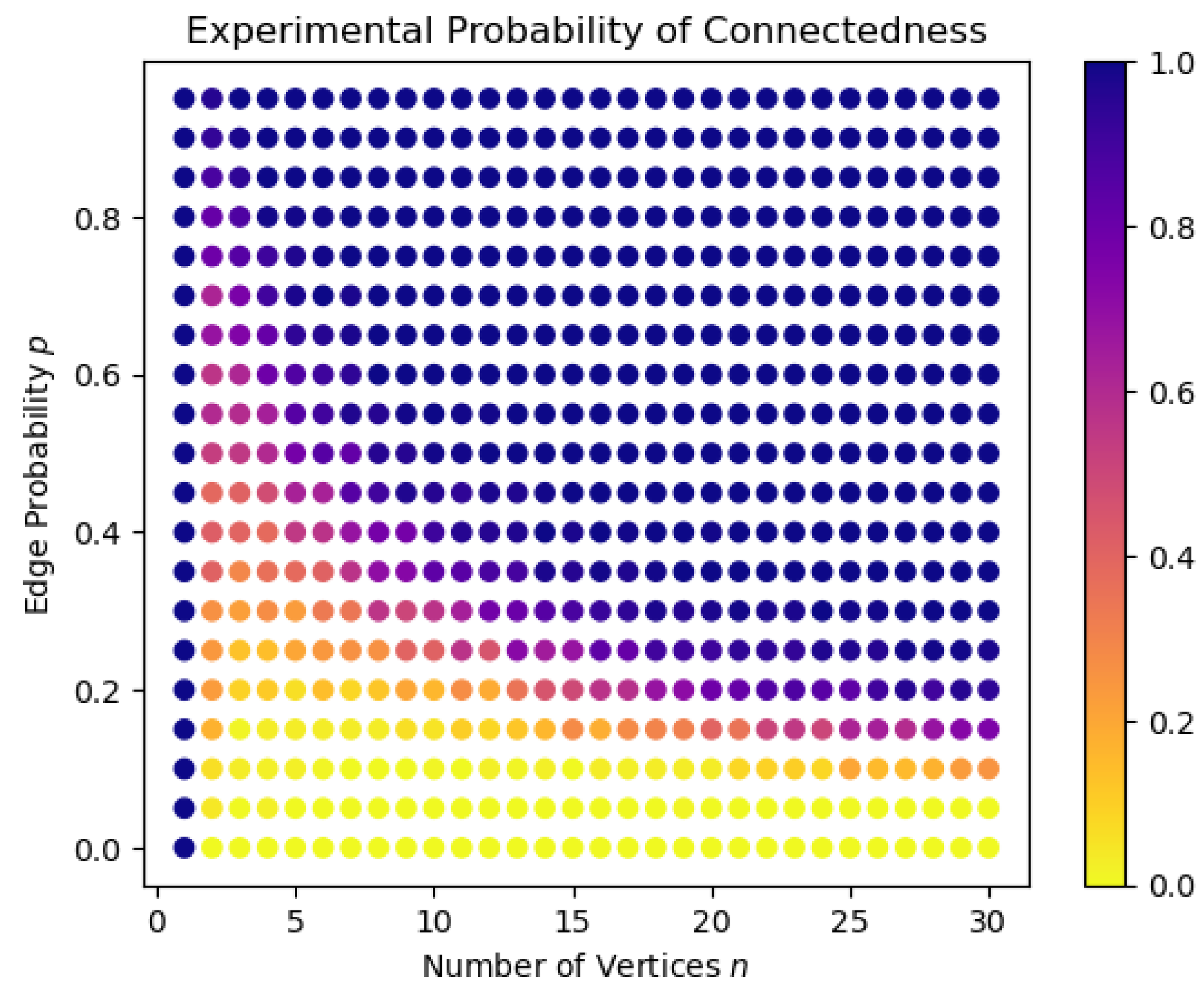
A connected graph and a disconnected graph

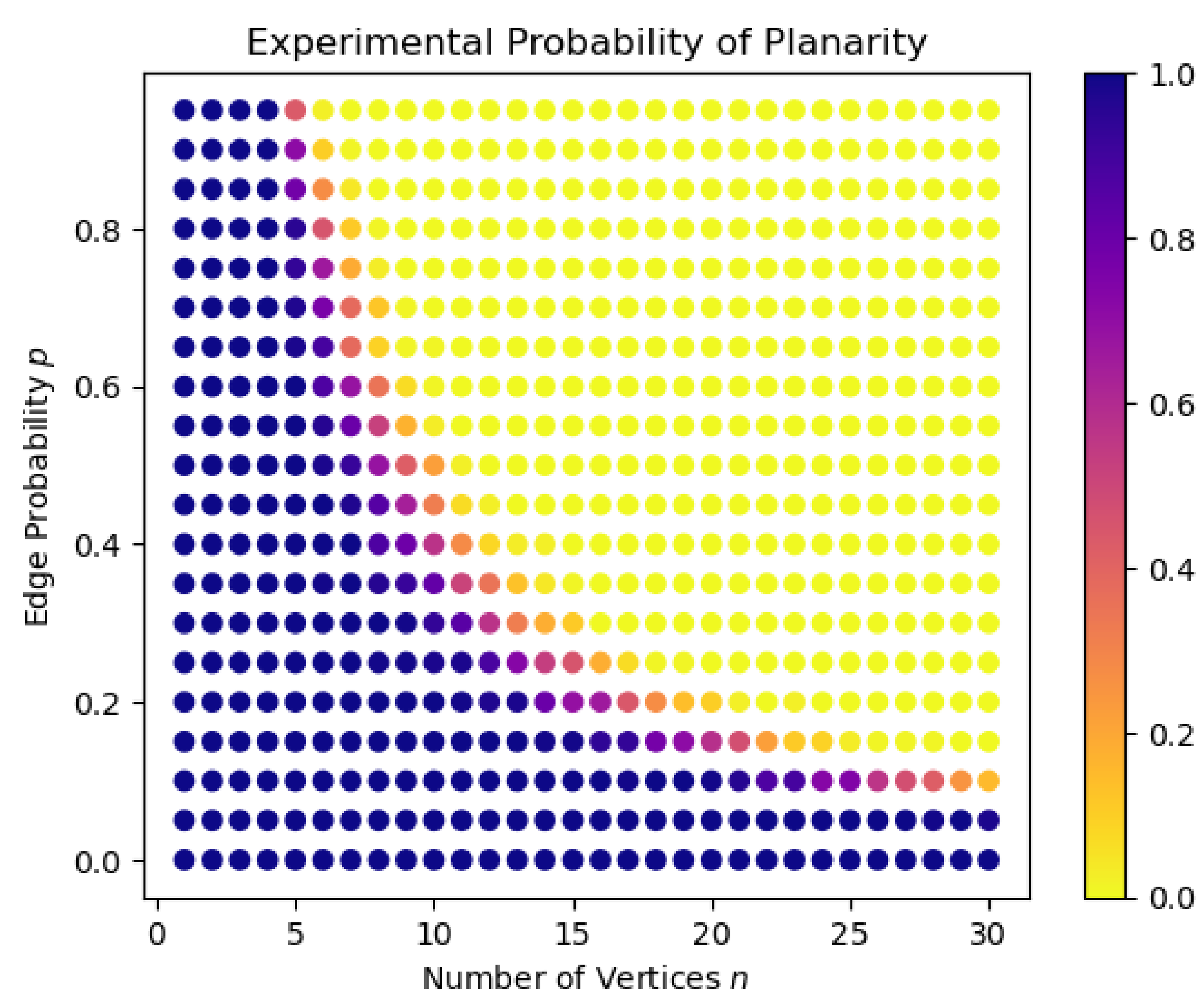
Definition 4. A graph G is *planar* if it is possible to draw G in the plane without any of its edges crossing.



Two important nonplanar graphs: K_5 and $K_{3,3}$

Results





Procedure

We use a combination of three algorithms to estimate the probability that a random graph G sampled from $G(n, p)$ is connected or planar.

Algorithm 1 generates the adjacency matrix of a random graph sampled from $G(n, p)$. Starting with an $n \times n$ matrix of all 0's, Algorithm 1 includes each edge with probability p.

Algorithm 2 determines connectedness using the standard and efficient technique known as breadth-first search (BFS). Using a BFS, Algorithm 2 explores as much of the graph as possible through adjacency, outputting a 1 if the whole graph can be explored (i.e. the graph is connected) and a 0 otherwise.

Algorithm 3 relies on a theorem of Kuratowski, which guarantees that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ (depicted in Definitions). Using a brute force search, Algorithm 3 outputs a 0 if any subset of 5 (resp. 6) vertices admits a K_5 (resp. $K_{3,3}$) subdivision and a 1 otherwise.

Using these algorithms, we run the following procedure to estimate the probability of connectedness (resp. planarity):

- Fix an integer m corresponding to the desired number of trials.
- Input the given values of n and p into Algorithm 1 to generate the adjacency matrices of m random graphs sampled from $G(n, p)$.
- Determine how many of these m graphs are connected (resp. planar) using Algorithm 2 (resp. 3).
- Divide this number by m to get the experimental probability of connectedness (resp. planarity).

Analysis

For each $n \in \{1, 2, 3, \ldots, 30\}$ and $p \in \{0.00, 0.05, 0.10, 0.15, \ldots, 2.00\}$, we used $m = 100$ trials to estimate probability of connectedness and planarity (see Results). In both figures, the x-axis represents the number of vertices n, and the y-axis describes the edge probability p. The warmer the color, the less likely to have the property.

For connectedness, as p increases, the more likely the graph will be connected due to the high probability of edges being formed. As for planarity, as p increases, the more edges will be formed and the more likely intersections will happen, so less planarity.

References

[1] Reinhard Diestel (2025) *Graph Theory*, Springer.