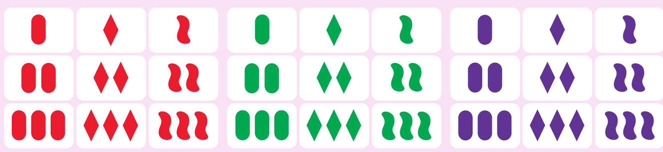
The Cap SET Problem



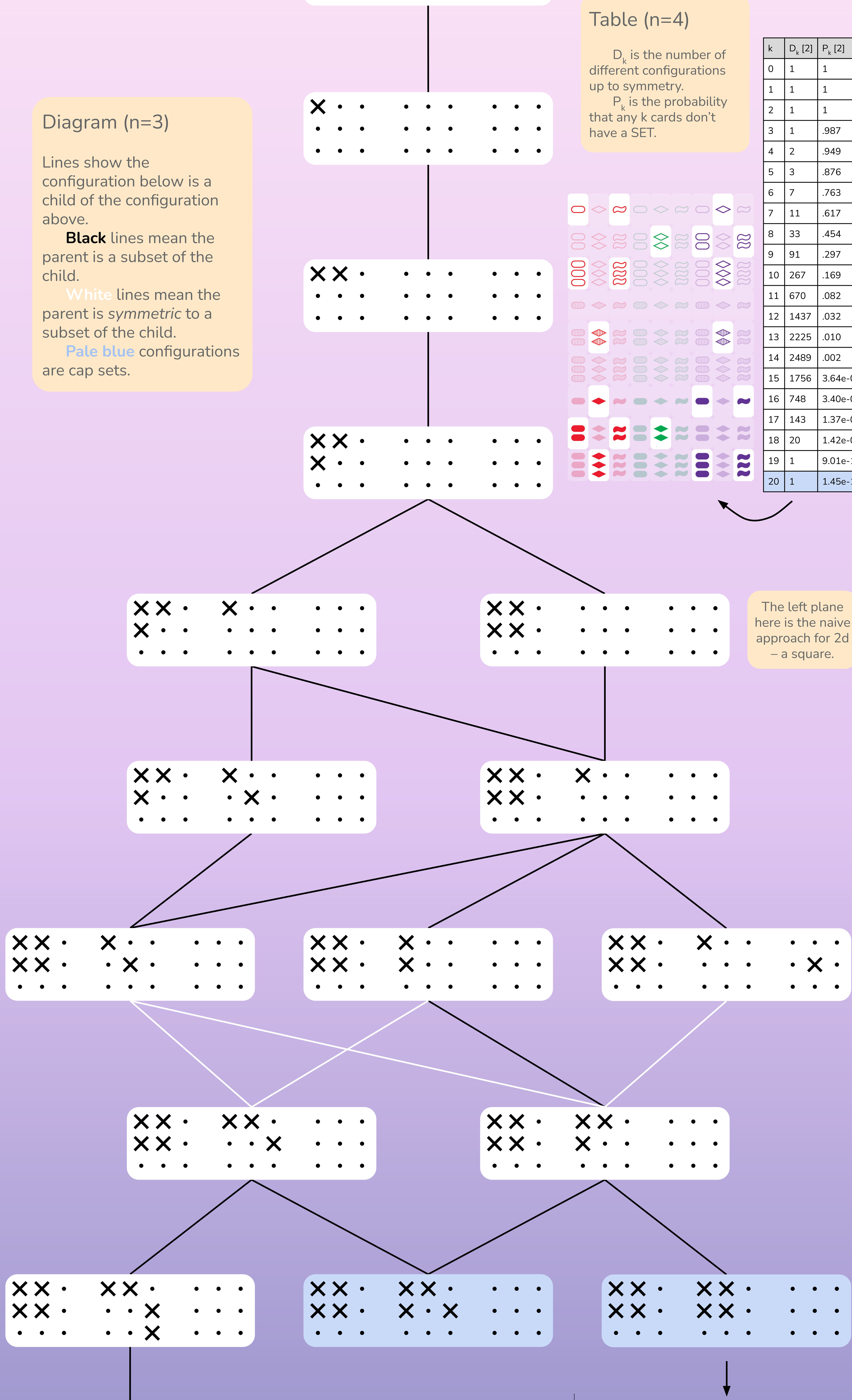


Diagram (n=3)

Lines show the configuration below is a child of the configuration above.

Black lines mean the parent is a subset of the child.

White lines mean the parent is *symmetric* to a subset of the child.

Pale blue configurations are cap sets.

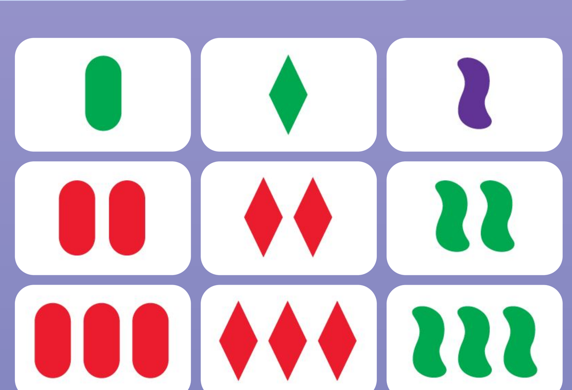
Table (n=4)

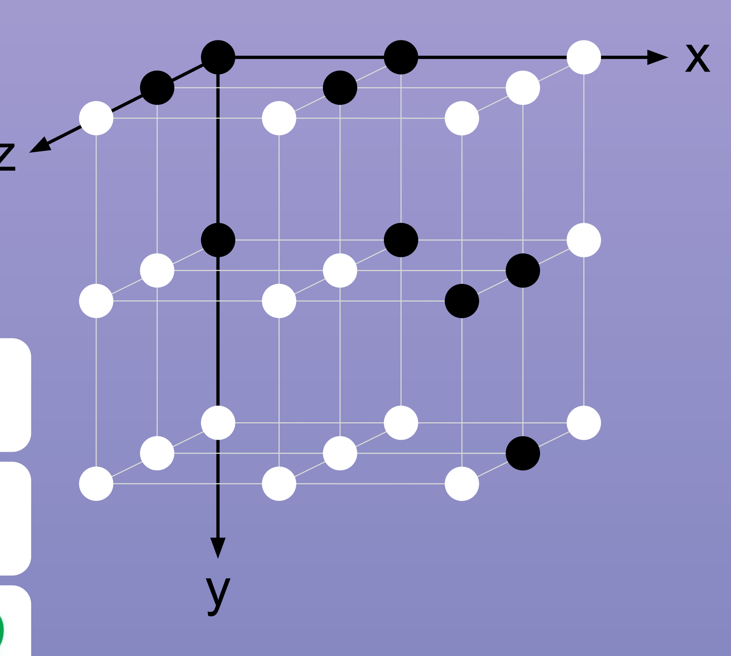
D_k is the number of different configurations up to symmetry.

P_k is the probability that any k cards don't have a SET.

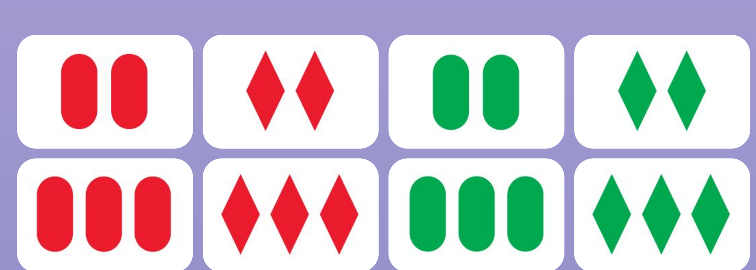
k	D_k [2]	P_k [2]
0	1	1
1	1	1
2	1	1
3	1	.987
4	2	.949
5	3	.876
6	7	.763
7	11	.617
8	33	.454
9	91	.297
10	267	.169
11	670	.082
12	1437	.032
13	2225	.010
14	2489	.002
15	1756	3.64e-04
16	748	3.40e-05
17	143	1.37e-06
18	20	1.42e-08
19	1	9.01e-12
20	1	1.45e-13

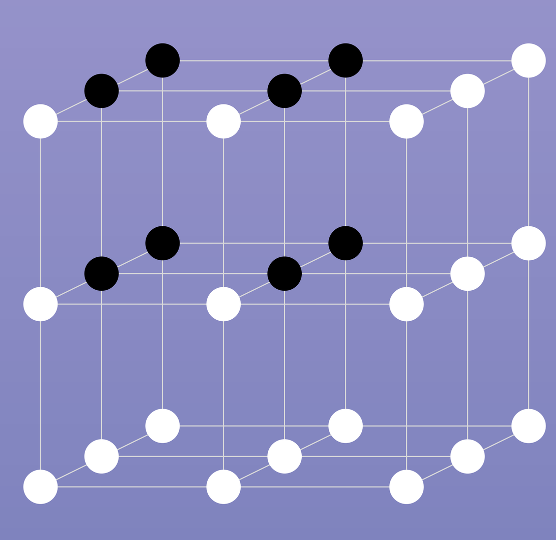
The left plane here is the naive approach for 2d – a square.





Solution to the 3d cap set problem





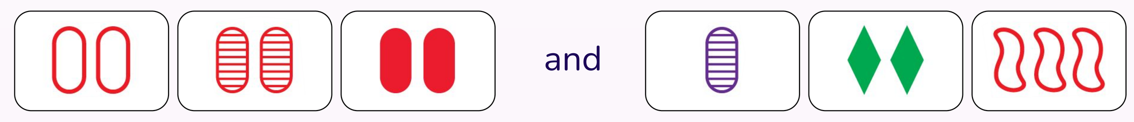
Naive Approach

The Game

SET is a card game with 81 cards that each have four attributes:

1. Color - red, green, or purple
2. Shape - oval, diamond, or squiggle
3. Number - one, two, or three
4. Shading - empty, striped, or solid

The goal is to find *SETs*: three cards that for each attribute they all either share the same value or all have different values. For example:



are both SETs. A *cap set* is a collection of cards that has no SETs and the addition of any other card will create a SET. The cap set problem asks:

What is the largest cap set in a deck with n attributes?

Symmetry

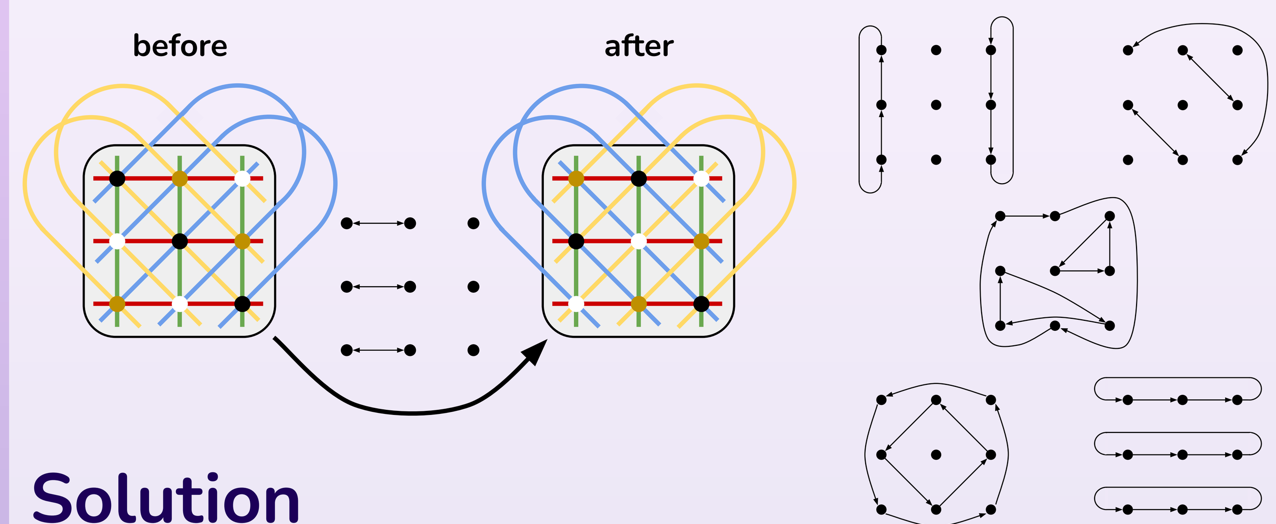
Each card can be uniquely defined by its specific values per attribute. Since each attribute ranges over three different values, each card can be represented as a list of four coordinates in the set {0, 1, 2}. Each value of an attribute can be labeled as the zeroth, first, or second value respectively. For example, look at the first card in the right SET. Its color is purple, which is the second value of the color attribute in the definition above. Next, its shape is an oval, which is the zeroth value of the shape attribute. The number of symbol is one, the zeroth value, and its shading is striped, the first value. Altogether this gives (2, 0, 0, 1). So, the SETs above can be rewritten as

(0, 0, 1, 0), (0, 0, 1, 1), (0, 0, 1, 2) and **(2, 0, 0, 1), (1, 1, 1, 2), (0, 2, 2, 0)**

Now, for any one coordinate, all three cards could have the same value, like how in the left SET the first coordinate of each card is 0, or they could all be different, like the fourth coordinate in the left SET. These can be interpreted as coordinates in 4d space mod 3, and it turns out three cards make a SET if and only if their corresponding points are collinear in this space. For example, let L(t)=at+b be a line parametrized by t in {0, 1, 2} with constant points a and b. The SETs above can then be written as

$L_1(t)$ = (0, 0, 0, 1)*t + (0, 0, 1, 0) and **$L_2(t)$ = (2, 1, 1, 1)*t + (2, 0, 0, 1)**

The "intercept" of each line is the first point, and the "slope" is the second point minus the first point. E.g. in the right SET, the slope is (1, 1, 1, 2) - (2, 0, 0, 1) = (-1, 1, 1, 1) = (2, 1, 1, 1). The cap set problem now asks **what is the maximum size of a collection of points in n dimensions where no three lie on a line?** One could try to check every possible combination of points, but for n=4 there are 2^{81} = 2.4 septillion possible combinations. Instead, there are symmetries of the points that can be taken advantage of. For example, adding 1 to the first coordinate of every point preserves lines because it is the same as adding 1 to the first coordinate of the intercept. Swapping the second and third coordinates in every point also preserves lines, as the second and third coordinates in the slope and intercept can be swapped. However, there are also some very unintuitive line-preserving symmetries. Below are some examples of symmetries in the 2d (n=2) case. All lines of the same color are parallel.



Solution

It turns out there are $3^4(3^4-1)(3^4-3)(3^4-3^2)(3^4-3^3)$ = 1965150720 symmetries in 4d (formally called the affine group over $\mathbb{Z}_3^4$). Only checking configurations up to these symmetries greatly reduces how many have to be checked. Define a *parent* configuration and *child* configuration such that the parent is symmetrical to the child minus a point. Then, one way to enumerate all configurations is with the following algorithm:

1. Start with an empty configuration of points.
2. For every configuration found so far, try to add a point to every position that won't create three in a line.
3. Check this new potential child arrangement is different compared to all previous arrangements up to symmetry.
    a. If it is, add it to the diagram with a line showing it is the child of its parent
    b. If it isn't, add a line from its parent to the configuration it is symmetrical to
4. Repeat step 2 until all configurations have been checked.

Plugging in n=3 gives the diagram to the left. The results for the original case, n = 4, are condensed in the table.

Explanation & Future Questions

A naive approach to generate cap sets is to consider the collection of all cards without any twos in their coordinates. This is a cap set because any two cards in this collection must differ at some coordinate, so the third card that forms a SET with them would have a two at that coordinate. This approach works for n=2, and for n=3 corresponds to the rightmost cap set in the diagram. However, this is not optimal. The lowest node in the diagram has nine points, meaning for n=3 it takes at least ten cards to ensure there is a SET.

For the original n=4, the algorithm finds 9908 different configurations before ending at a configuration of size 20, an instance of which is left of the table. In both the diagram and the table, the maximal cap set only has one parent. This means regardless of the point removed, all resulting configurations are symmetrical. Formally, this property is called "vertex-transitivity".

In general, it is difficult to both find a largest cap set and prove there are none larger. The sequence of known sizes of maximal cap sets starting from n = 0 goes 1, 2, 4, 9, 20, 45, 112, ... [1]. As of April 2025, values for n > 6 are unknown. Also, for each of these cases so far there is only one maximal cap set up to symmetry, even though there is no reason this should be the case.

More Information

[1] https://oeis.org/A090245
[2] https://www.ams.org/publicoutreach/feature-column/fc-2015-08
[3] https://www.dcode.fr/set-solver
[4] https://en.wikipedia.org/wiki/Cap_set

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