2024 Algebra 6112 Qualifying Exam
Instructions

Enter your name.# on the roster sheet together with a code name for yourself that is different from any code name that has already been entered.

Answer each question on a separate sheet of paper, and write your code name and the problem number on each sheet of paper that you submit for grading. Do not put your real name on any sheet of paper that you submit for grading.

Answer as many questions as you can. Do not use theorems which make the solution to the problem trivial. Always clearly display your reasoning. The judgment that you use in this respect is an important part of the exam.

This is a closed book, closed notes exam.

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1. (a) Show that if a functor $F\colon \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is faithful, then for any morphism f in $\mathcal{C}$, if $F(f)$ is a monomorphism then f is a monomorphism.

   (b) On the other hand, give an example of a faithful functor $F\colon \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ (of your choosing) and a monomorphism f in $\mathcal{C}$ such that $F(f)$ is not a monomorphism.

2. Let R be a ring. Prove that a left R-module Q is injective if and only if for any left ideal $\mathfrak{a}$ of R, any homomorphism $\mathfrak{a} \to Q$ can be extended to a homomorphism $R \to Q$. (Hint for the substantial direction: for any inclusion $N \subset M$ of R-modules and homomorphism $f\colon N \to Q$, use Zorn's lemma to produce a maximal extension of f inside M, and then argue by contradiction.)

3. If p and q are primes (not necessarily distinct), compute $\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/q)$ for all $n \geq 0$.

4. Let E/F be an algebraic extension. Let $\sigma : E \to E$ be a morphism of fields such that $\sigma|_F = \mathrm{Id}_F$. Prove that σ is then an isomorphism.

5. Let K be the splitting field of $x^4 - 2$ over $\mathbb{Q}$. Compute the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.