Real Analysis Qualifying Examination 1 **Monday, August 12, 2024**

Please start each problem on a new page and remember to write your code on each page of your answers.

You should exercise good judgement in deciding what constitutes an adequate solution. In particular, you should not try to solve a problem by just quoting a theorem that reduces what you are asked to prove to a triviality. If you are not sure whether you may use a particular theorem, ask the proctor.

[16] **1.** Let $A \subseteq [0,1]$ such that $m^*(A) + m^*(A^c) = 1$, where m^* is Lebesgue outer measure on $[0,1]$ and $A^c = [0,1] \setminus A$. Prove that A is Lebesgue-measurable.

[17] **2.** Let X be a second countable topological space, let $\mathscr{B}$ be the Borel σ-algebra on X, and let μ be a measure on $\mathscr{B}$. Prove that there is a closed set $S \subseteq X$ such that $\mu(X \setminus S) = 0$ and for each closed proper subset $F \subseteq S$, $\mu(S \setminus F) > 0$.

3. Suppose f is a non-negative Lebesgue-integrable function on $[0,1]$.

[5] **(a)** Show that for each $n \in \mathbb{N}$, $\sqrt[n]{f}$ is Lebesgue-integrable.

[12] **(b)** Show that the sequence $(\sqrt[n]{f})$ converges in $L^1[0,1]$ and compute its limit.

Hint for both parts: Consider $\{ f > 1 \}$ *and* $\{ f \leq 1 \}$ *separately.*

[16] **4.** Let $(X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$ be σ-finite measure spaces, let $\mathscr{L} = \mathscr{M} \otimes \mathscr{N}$ be the product σ-algebra, and let $\lambda = \mu \otimes \nu$ be the product measure. For each $E \subseteq X \times Y$, for each $x \in X$, let

$$E_x = \{ y \in Y : (x, y) \in E \}.$$

Suppose $E \in \mathscr{L}$ such that $\lambda(E) = 0$. Show that $\nu(E_x) = 0$ for μ-a.e. $x \in X$.

[17] **5.** Let $f \in L^1[0,1]$ and suppose f is left-continuous at 1. Find

$$\lim_{n \to \infty} \int_0^1 n x^n f(x)\, dx.$$

Prove your answer.

6. Let μ and ν be finite signed measures on the measurable space $(X, \mathscr{M})$.

[8] **(a)** Prove that the signed measure $\mu \wedge \nu := \frac{1}{2}(\mu + \nu - |\mu - \nu|)$ satisfies

$$(\mu \wedge \nu)(E) \leq \min\{ \mu(E), \nu(E) \}$$

for all $E \in \mathscr{M}$.

[9] **(b)** Suppose in addition that μ and ν are positive. Prove that $\mu \perp \nu$ if and only if $\mu \wedge \nu = 0$.