Real Analysis Qualifying Examination 2 **Tuesday, August 13, 2023**

Please start each problem on a new page and remember to write your code on each page of your answers.

You should exercise good judgement in deciding what constitutes an adequate solution. In particular, you should not try to solve a problem by just quoting a theorem that reduces what you are asked to prove to a triviality. If you are not sure whether you may use a particular theorem, ask the proctor.

[16] **1.** Let E be a normed linear space and let S be a linear operator on E whose graph is closed in $E \times E$. Does it follow that S is continuous? Either prove that it does or give a counterexample.

[17] **2.** Let E be a normed linear space, let $A \subseteq E$, and suppose $f[A]$ is bounded in the scalar field for each continuous linear functional f on E. Prove that A is bounded in E.

[16] **3.** Let m be Lebesgue measure on $\mathbb{R}^d$, let E be a Lebesgue-measurable subset of $\mathbb{R}^d$ with $m(E) < \infty$, let $p \in [1, \infty)$, and let $L^p = L^p(m)$. Show that the characteristic function χ_E can be approximated in L^p by elements of $C_c(\mathbb{R}^d)$. (Warning: You should not attempt to solve this problem by using facts about convolutions, as that would be a circular argument. Those facts depend on the very thing you are being asked to prove here.)

4. Define $\tau_y f(x) = f(x-y)$ for f on $\mathbb{R}^d$ and $x, y \in \mathbb{R}^d$. Let $L^p = L^p(\mathbb{R}^d)$ for $1 \le p \le \infty$.

[13] **(a)** Suppose $f \in L^p$, where $1 \le p < \infty$. Prove that

$$\lim_{y \to 0} \|\tau_y f - f\|_{L^p} = 0.$$

[4] **(b)** Does the conclusion of part (a) hold in the case where $p = \infty$?

[17] **5.** Suppose f, f', f'' are in $L^2(\mathbb{R})$. Show that $\widehat{f'} \in L^1(\mathbb{R})$.

[8] **6. (a)** Show that for each $\varphi \in C_c^\infty(-1, 1)$, the limit

$$\lim_{\varepsilon \to 0} \int_{\varepsilon < |x| < 1} \frac{\varphi(x)}{x}\, dx \qquad (*)$$

exists in $\mathbb{C}$.

[9] **(b)** Define F on $C_c^\infty(-1, 1)$ by letting $F(\varphi)$ be the limit $(*)$ for each $\varphi \in C_c^\infty(-1, 1)$. Show that F is a distribution in $(-1, 1)$.