Real Analysis Qualifying Examination 1 **Monday, August 18, 2025**

Please start each problem on a new page and remember to write your code on each page of your answers.

You should exercise good judgement in deciding what constitutes an adequate solution. In particular, you should not try to solve a problem by just quoting a theorem that reduces what you are asked to prove to a triviality. If you are not sure whether you may use a particular theorem, ask the proctor.

[16] **1.** Let $(X, \mathscr{A}, \mu)$ be a measure space. Suppose μ is semifinite.[1] Prove that for each $B \in \mathscr{A}$, if $\mu(B) = \infty$, then for each $t < \infty$, there exists $A \in \mathscr{A}$ such that $A \subseteq B$ and $t < \mu(A) < \infty$.

[17] **2.** Let (X, ρ) be a separable metric space, let $\mathscr{B}$ be the Borel σ-algebra on X, let μ be a finite measure on $\mathscr{B}$, and let $\varepsilon > 0$. Prove that there is a closed totally bounded[2] set $A \subseteq X$ such that $\mu(X \setminus A) < \varepsilon$.

[17] **3.** Let $(X, \mathscr{A}, \mu)$ be a measure space. Let $f, f_1, f_2, \ldots : X \to \mathbb{R}$ be measurable. Suppose $f_n \to f$ pointwise and $\int |f_n| \, d\mu \to \int |f| \, d\mu < \infty$. Prove that $\int |f - f_n| \, d\mu \to 0$.

[17] **4.** Let m be Lebesgue measure on $\mathbb{R}^n$ and let f be a nonnegative Lebesgue-measurable function on $\mathbb{R}^n$. Prove that

$$\int_{\mathbb{R}^n} f \, dm = \int_0^\infty m(f > y) \, dy,$$

where $m(f > y)$ denotes $m(\{ x \in \mathbb{R}^n : f(x) > y \})$.

[17] **5.** Suppose $f : \mathbb{R} \to \mathbb{C}$ is continuous, $g : \mathbb{R} \to \mathbb{C}$ is Lebesgue-integrable, and for each compactly supported infinitely differentiable function $\varphi : \mathbb{R} \to \mathbb{C}$, we have

$$\int_{\mathbb{R}} f(x)\varphi'(x) \, dx = -\int_{\mathbb{R}} g(x)\varphi(x) \, dx.$$

Prove that f is absolutely continuous and that $f' = g$ almost everywhere.

[16] **6.** Show that any two norms on $\mathbb{R}^n$ are equivalent.[3]

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[1] To say that μ *is semifinite* means that for each $B \in \mathscr{A}$, if $\mu(B) = \infty$, then there exists $A \in \mathscr{A}$ such that $A \subseteq B$ and $0 < \mu(A) < \infty$.

[2] To say that A *is totally bounded* means that for each $\delta > 0$, A can be covered by finitely many balls of radius δ.

[3] To say that two norms are *equivalent* means that each is bounded by a constant multiple of the other.