Real Analysis Qualifying Examination 2 **Tuesday, August 19, 2025**

Please start each problem on a new page and remember to write your code on each page of your answers.

You should exercise good judgement in deciding what constitutes an adequate solution. In particular, you should not try to solve a problem by just quoting a theorem that reduces what you are asked to prove to a triviality. If you are not sure whether you may use a particular theorem, ask the proctor.

[16] **1.** Let L^p denote $L^p(\mathbb{R}^d)$ for each $p \in [1, \infty]$. Let $1 \le p \le \infty$. Let $f \in L^p$ and let $g \in L^1$. Prove that $\|f * g\|_{L^p} \le \|f\|_{L^p} \|g\|_{L^1}$.

[16] **2.** Prove that the operator T on $L^2(0, \infty)$ defined by

$$Tf(x) = \frac{1}{\pi} \int_0^\infty \frac{f(y)}{x+y}\, dy$$

is bounded with $\|T\| \le 1$.

[17] **3.** Let X, Y, Z be Banach spaces, and let $B : X \times Y \to Z$ be linear and continuous in each variable separately. More precisely, suppose for each fixed $y \in Y$, the function $B(\cdot, y)$ is a continuous linear map $X \to Z$, and similarly for fixed $x \in X$ and $B(x, \cdot)$. Prove that B is continuous.

[17] **4.** Let X be a locally compact Hausdorff space. Let $C_0(X)$ be the vector space of continuous real-valued functions on X which tend to zero at infinity. As usual, give $C_0(X)$ the uniform norm, defined by

$$\|f\|_u = \sup\{\, |f(x)| : x \in X \,\}$$

for each $f \in C_0(X)$. Let L be a linear functional on $C_0(X)$ such that for each $f \in C_0(X)$, if $f \ge 0$, then $L(f) \ge 0$. Prove that L is continuous.

[17] **5.** Let $f : \mathbb{R} \to \mathbb{C}$ be 1-periodic and continuously differentiable. For each $k \in \mathbb{Z}$, let

$$c_k = \int_0^1 e^{-2\pi i k x} f(x)\, dx.$$

Prove that $\sum_{k\in\mathbb{Z}} |c_k| < \infty$.

[17] **6.** State and prove the Riemann-Lebesgue lemma for the Fourier transform on $\mathbb{R}$.