Solutions to 2026 Gordon examination problems

1. *Let p be a nonconstant polynomial with integer coefficients; prove that the number $\overline{0.p(1)p(2)\dots}$ (the concatenation of the decimal expansions of the integers $p(1)$, $p(2)$, etc.) is irrational.*

Solution. The formulation is inaccurate: it is not clear what is meant if p takes negative values. It should be added that if $p(n)$ is negative, the "$-$" sign should be ignored. (In other words, we consider the concatenation of the decimal expansions of $|p(n)|$.)

Let $p(n) = a_d n^d + \dots + a_1 n + a_0$ where $a_d \neq 0$. First, assume that $a_d > 0$ and $a_0 \geq 0$. Let $\overline{c_1 \dots c_k}$ be the decimal expansion a_0. and let n_0 be such that $p(n) > a_0$ for all $n > n_0$. Then for any r such that $r > k$ and $10^r > n_0$ the decimal expansion of $p(10^r)$ is $\overline{d_1 \dots d_n 0 \dots 0 c_1 \dots c_k}$ with a block of at least $r-k$ zeroes. It follows that the sequence $\overline{p(1)p(2)\dots}$ contains arbitrarily long blocks of 0s; this cannot happen if the corresponding number is rational (since in this case the sequence of its digits would be eventually periodic).

The case $a_d < 0$ and $a_0 \leq 0$ is obtained by replacing p by $-p$. If a_d and a_0 have opposite signs, say $a_d > 0$ and $a_0 < 0$, then for r large enough the expansions of $p(10^r)$ contain arbitrarily long sequnces of 9s.

2. *A point (n,m) of $\mathbb{Z}^2$ is said to be primitive if n and m are coprime integers. Prove that there are arbitrarily large disks in $\mathbb{R}^2$ containing no primitive points.*

Solution. Fix $N \in \mathbb{N}$. Choose N^2 distinct primes $p_{i,j}$, $1 \leq i,j \leq N$. For every i let $n_i = \prod_{j=1}^N p_{i,j}$ and for every j let $m_j = \prod_{i=1}^N p_{i,j}$. Since all n_i are pairwise coprime and all m_j are pairwise coprime, by the Chinese remainder theorem there exist $n \in \mathbb{Z}$ such that for every i, $n = -i \bmod n_i$, and $m \in \mathbb{Z}$ such that for every j, $m = -j \bmod m_j$. Then for every $i,j \in \{1,\dots,N\}$, $n+i$ is divisible by n_i and $m+j$ is divisible by m_j, so both $n+i$ and $m+j$ are divisible by $p_{i,j}$. Hence all the points of the $N \times N$ square $\{n+1,\dots,n+N\} \times \{m+1,\dots,m+N\}$ are non-primitive.

3. *Suppose the polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1 x + 1$ with $a_1,\dots,a_{n-1} \geq 0$ has n real roots. Prove that $p(2) \geq 3^n$.*

Solution. Since all the coefficients of p are nonnegative, $p(x) > 0$ for all $x \geq 0$, so all the roots of p are negative and $p(x) = \prod_{i=1}^n (x+\alpha_i)$ with $\alpha_1,\dots,\alpha_n > 0$. Since $p(0) = 1$ we have $\alpha_1 \cdots \alpha_n = 1$. For every i, $2+\alpha_i = 1+1+\alpha_i \geq 3\sqrt[3]{\alpha_i}$; so $p(2) = \prod_{i=1}^n (2+\alpha_i) \geq 3^n \prod_{i=1}^n \sqrt[3]{\alpha_i} = 3^n \sqrt[3]{\prod_{i=1}^n \alpha_i} = 3^n$.

4. *Prove that* $\displaystyle\int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x}\,dx = \begin{cases} 0, & n \text{ is even} \\ \pi, & n \text{ is odd.} \end{cases}$

Solution. For any n the function $\frac{\sin(nx)}{(1+2^x)\sin x}$ is continuous on $(-\pi,\pi)$ with the limits $\frac{n}{1+2^{-\pi}}$ and $\frac{n}{1+2^{\pi}}$ as $x \longrightarrow -\pi$ and π respectively, thus is integrable on $[-\pi,\pi]$. Let $I_n = \int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x}\,dx$, $n \geq 0$. Since $\frac{1}{1+2^{-x}} = \frac{2^x}{1+2^x}$,

$$I_n = \int_{-\pi}^{0} \frac{\sin(nx)}{(1+2^x)\sin x}\,dx + \int_0^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x}\,dx = \int_0^{\pi} \frac{\sin(-nx)}{(1+2^{-x})\sin(-x)}\,dx + \int_0^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x}\,dx$$
$$= \int_0^{\pi} \frac{2^x \sin(nx)}{(1+2^x)\sin(x)}\,dx + \int_0^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x}\,dx = \int_0^{\pi} \frac{(1+2^x)\sin(nx)}{(1+2^x)\sin(x)}\,dx = \int_0^{\pi} \frac{\sin(nx)}{\sin x}\,dx.$$

For any $n \geq 2$, $\sin(nx) - \sin((n-2)x) = 2\cos((n-1)x)\sin x$, so

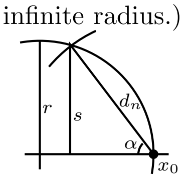
$$I_n = \int_0^{\pi} \frac{\sin(nx)}{\sin x}\,dx = \int_0^{\pi} \frac{\sin((n-2)x)}{\sin x}\,dx + 2\int_0^{\pi} \cos((n-1)x)\,dx = I_{n-2} + 0 = I_{n-2}.$$

Hence, if n is even, $I_n = I_0 = 0$; if n is odd, $I_n = I_1 = \int_0^{\pi} dx = \pi$.

5. *Suppose that each point of $\mathbb{R}^3$ is colored with one of 3 colors. Prove that the pairs of points in at least one of these colors achieve all positive distances.*

Solution. We will prove this more generally for $\mathbb{R}^n$ colored with n colors. Moreover, to use induction, we will be proving a similar statement not only for $\mathbb{R}^n$ but also for any n-dimensional sphere; namely, that if the points of an n-dimensional sphere S of radius r are colored with n colors then for every $d \le r\sqrt{2}$ there are $x_1, x_2 \in S$ such that $\operatorname{dist}(x_1, x_2) = d$. (We can consider $\mathbb{R}^n$ as an n-dimensional sphere of infinite radius.)

The case $n = 1$ is obvious, so we assume that $n \ge 2$. Let S be colored with the colors $C_1, \dots, C_n$ and let $X_i = \{x \in S \text{ colored } C_i\}$, $i = 1, \dots, n$. Suppose that for every i there is $0 < d_i \le r\sqrt{2}$ (or just $d_i > 0$ if $S_n = \mathbb{R}^n$) such that there are no $x_1, x_2 \in X_i$ with $\operatorname{dist}(x_1, x_2) = d_i$. W.l.o.g. assume that $X_n \ne \emptyset$ and $d_n \ge d_i$ for all i such that $X_i \ne \emptyset$. Choose $x_0 \in X_n$ and let $P = \{x \in S : \operatorname{dist}(x, x_0) = d_n\}$, then P is an $(n-1)$-dimensional sphere of radius $s \ge d_n/\sqrt{2}$. Since x_0 has color C_n the points of P are colored with the $n-1$ colors $C_1, \dots, C_{n-1}$. By induction on n for some $i \le n-1$ for every $d \le s\sqrt{2}$ there are $x_1, x_2 \in P \cap X_i$ with $\operatorname{dist}(x_1, x_2) = d$; in particular, this is true for $d = d_i$, which contradicts the choice of d_i.



$\alpha \ge \pi/4$, so $s = d_n \sin\alpha \ge d_n/\sqrt{2}$

6. *Prove that any nonzero real $n \times n$-matrix A with $n \ge 2$ can be represented as $A = B + C$ with $\det B = \det C = 1$.*

Solution. The proof works for any field, not necessarily $\mathbb{R}$.

The operation of adding a multiple of one column or row to another column or, respectively, row doesn't change the determinant and is additive (commutes with the addition of matrices). Hence, the statement holds for a matrix A iff it holds for any matrix A' obtained from A by a sequence of those row and column operations. Using these operations we can transform A to a matrix with only 2-s on the main diagonal,

$$A' = \begin{pmatrix} 2 & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ a_{2,1} & 2 & a_{2,3} & \dots & a_{2,n} \\ a_{3,1} & a_{3,2} & 2 & \dots & a_{3,n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \dots & 2 \end{pmatrix}, \text{ and put } B = \begin{pmatrix} 1 & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ 0 & 1 & a_{2,3} & \dots & a_{2,n} \\ 0 & 0 & 1 & \dots & a_{3,n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ a_{2,1} & 1 & 0 & \dots & 0 \\ a_{3,1} & a_{3,2} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \dots & 1 \end{pmatrix}.$$

The procedure of transforming a nonzero square matrix A to a matrix with only 2-s on the main diagonal can be as follows. Choose i, j such that the (i, j)-th entry of A is nonzero. By adding suitable multiples of the i-th row of A to the other rows make all the entries in the j-th column nonzero and, if $i \ne j$, the (j, j)-th entry equal to 2; in the case $i = j$ add a multiple of another row to the i-th row to make the (j, j)-th entry equal to 2. Then add suitable multiples of the j-th column of A to the other columns to make all other diagonal entries equal to 2 too.