Solutions to 2026 Rasor-Bareis examination problems

1. *Prove that the number* $0.1234567891011121314\ldots$ *is irrational.*

Solution. The decimal expansion $0.d_1d_2d_3\ldots$ of any rational number is eventually periodic: there are $n, k \in \mathbb{N}$ such that $d_{m+k} = d_m$ for all $m \geq n$. Thus it contains only finitely many different length k subsequences. However the sequence $123456789101112\ldots$ contains every finite sequence of digits infinitely many times.

2. *Let* $n \geq 4$ *and let* $a_1, \ldots, a_n$ *be positive real numbers with* $a_1 \cdots a_n = 1$. *Prove that*

$$\frac{1}{1+a_1+a_1a_2} + \frac{1}{1+a_2+a_2a_3} + \cdots + \frac{1}{1+a_n+a_na_1} > 1.$$

Solution. Put $b_1 = a_1$, $b_2 = a_1a_2$, $\ldots$, $b_{n-1} = a_1a_2\cdots a_{n-1}$, and $b_n = a_1a_2\cdots a_{n-1}a_n = 1$, so that $a_i = b_i/b_{i-1}$ for all $i = 2, \ldots, n$ and $a_1 = b_1/b_n$. Put $b = b_1 + \ldots + b_n$. Then for every $i = 2, \ldots, n-1$,

$$\frac{1}{1+a_i+a_ia_{i+1}} = \frac{1}{1+b_i/b_{i-1}+b_{i+1}/b_{i-1}} = \frac{b_{i-1}}{b_{i-1}+b_i+b_{i+1}} > \frac{b_{i-1}}{b},$$

and also $\frac{1}{1+a_1+a_1a_2} = \frac{1}{1+b_1+b_2} = \frac{b_n}{b_n+b_1+b_2} > \frac{b_n}{b}$ and $\frac{1}{1+a_n+a_na_1} = \frac{1}{1+b_n/b_{n-1}+b_1/b_{n-1}} = \frac{b_{n-1}}{b_{n-1}+b_n+b_1} > \frac{b_{n-1}}{b}$. Adding these n inequalities we obtain

$$\frac{1}{1+a_1+a_1a_2} + \frac{1}{1+a_2+a_2a_3} + \cdots + \frac{1}{1+a_n+a_na_1} > \frac{b_n+b_1+b_2+\cdots+b_{n-1}}{b} = 1.$$

3. *Find the limit of the sequence of the fractional parts of* $(2+\sqrt{2})^n$, $n = 1, 2, \ldots$

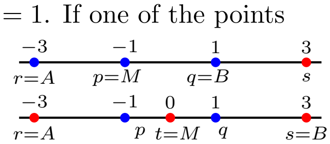
Solution. Let $x_n = (2+\sqrt{2})^n$ and $y_n = (2-\sqrt{2})^n$, $n = 1, 2, \ldots$ By the binomial formula, for any n

$$x_n + y_n = \sum_{i=0}^{n} \tbinom{n}{i} 2^{n-i}(\sqrt{2})^i + \sum_{i=0}^{n} \tbinom{n}{i} 2^{n-i}(-1)^i(\sqrt{2})^i = 2 \sum_{\substack{0\leq i\leq n \\ i \text{ is even}}} \tbinom{n}{i} 2^{n-i}(\sqrt{2})^i$$

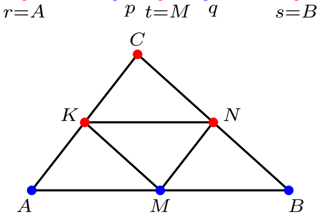
is an integer (since $(\sqrt{2})^i$ is an integer for every even i). Since, for any n, x_n and y_n are not integer, this implies that $\{x_n\} + \{y_n\} = 1$ (where $\{a\}$ denotes the fractional part of a). Since $0 < 2-\sqrt{2} < 1$, $0 < y_n < 1$ for all n and $y_n \longrightarrow 0$; hence, $\{y_n\} = y_n \longrightarrow 0$, and respectively $\{x_n\} \longrightarrow 1$.

4. *Given a triangle* T, *prove that for any 2-coloring of the points of the plane there are three points of the same color that form a triangle similar to* T.

Solution. Suppose the colors are red and blue. First, find two points A and B such that A, B, and the middle point M of the interval $[A, B]$ have same color. For this end, take two points p and q of same color (say, blue) and introduce coordinates on the line (p, q) such that $p = -1$ and $q = 1$. If one of the points $r = -3$ or $s = 3$ of this line is also blue, take A, M, B to be r, p, q or p, q, s respectively. And if both r and s are red, if the point $t = 0$ is blue take as A, M, B the points p, t, q; if t is red, take r, t, s.



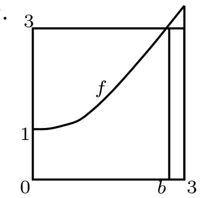
We now have blue points A, B and M such that M is the middle point of $[A, B]$. Choose a point C such that $\triangle ABC$ is similar to T. If C is also blue, we are done. Suppose that C is red. Let K be the middle point of $[AC]$ and N be the middle point of $[BC]$. If K is blue, then $\triangle AMK$ is blue (has all vertices blue) and similar to T; if N is blue, then $\triangle MBN$ is blue and similar to T; if both K and N are red, then $\triangle KNC$ is red and similar to T.



5. *Prove that* $\int_0^3 \sqrt[4]{x^4+1}\,dx + \int_1^3 \sqrt[4]{x^4-1}\,dx > 9.$

Solution. The function $g(y) = \sqrt[4]{y^4-1}$ on $[1,+\infty)$ is the inverse of the function $f(x) = \sqrt[4]{x^4+1}$ on $[0,+\infty)$, with $g(1) = 0$ and $g(3) = b$ where $b = \sqrt[4]{3^4-1} < 3$. Substituting y by $f(x)$ and integrating by parts we obtain:

$$\int_1^3 g(y)\,dy = \int_0^b x f'(x)\,dx = x f(x)\Big|_0^b - \int_0^b f(x)\,dx = b f(b) - \int_0^b f(x)\,dx = 3b - \int_0^b f(x)\,dx.$$



Since $f(x) > 3$ on the interval $(b, 3]$, we also have $\int_b^3 f(x)\,dx > 3(3-b) = 9 - 3b$, so

$$\int_1^3 g(y)\,dy + \int_0^3 f(x)\,dx = \int_1^3 g(y)\,dy + \int_0^b f(x)\,dx + \int_b^3 f(x)\,dx > 3b + 9 - 3b = 9.$$

6. *Find all functions* $f\colon \mathbb{R}\setminus\{0,1\} \longrightarrow \mathbb{R}$ *satisfying* $f(1/x) + f(1-x) = x$ *for all* $x \neq 0, 1$.

Solution. After replacing x by $1/x$, by $1-x$, and by $x/(x-1)$ the identity $f(1/x) + f(1-x) = x$ becomes, respectively:

$$f(x) + f\big(1 - \tfrac{1}{x}\big) = \tfrac{1}{x}, \quad f\big(\tfrac{1}{1-x}\big) + f(x) = 1 - x, \quad \text{and} \quad f\big(1 - \tfrac{1}{x}\big) + f\big(\tfrac{1}{1-x}\big) = \tfrac{x}{x-1}.$$

After subtracting the third identity from the sum of the first two we obtain $2f(x) = \frac{1}{x} + 1 - x - \frac{x}{x-1}$.