2025 Algebra 6111 Qualifying Exam
Instructions

Enter your name.# on the roster sheet together with a code name for yourself that is different from any code name that has already been entered.

Answer each question on a separate sheet of paper, and write your code name and the problem number on each sheet of paper that you submit for grading. Do not put your real name on any sheet of paper that you submit for grading.

Answer as many questions as you can. Do not use theorems which make the solution to the problem trivial. Always clearly display your reasoning. The judgment that you use in this respect is an important part of the exam.

This is a closed book, closed notes exam.

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1. Prove the following special case of Cauchy's theorem: If G is a finite abelian group of order n, and p is a prime dividing n, then G contains an element of order p.

2. Let H be a subgroup of a finite group G such that every element in G is conjugate to some element in H. Show that $H = G$.

3. Let R be a commutative ring with 1 and let M be an R-module. For a prime ideal $\mathfrak{p} \subset R$, let $M_{\mathfrak{p}}$ be the localization at the multiplicatively closed set $S = R \setminus \mathfrak{p}$. Prove that
$$M = 0 \text{ if and only if } M_{\mathfrak{p}} = 0 \text{ for every prime ideal } \mathfrak{p}.$$

4. Prove *Sylvester's Law of Inertia*: Any real symmetric matrix A is cogredient to a diagonal matrix D, and the numbers of positive, negative, and zero diagonal entries of D is an invariant of A—that is, if D' is another diagonal matrix cogredient to A, then these numbers are the same. (Two matrices A and B are *cogredient* if there is an invertible matrix P such that $P^t AP = B$.)

5. Let R be a commutative Noetherian ring with 1, $S \subset R$ a subring. Must S be noetherian? If $Q = R/I$ is a quotient ring, must Q be Noetherian? (In each case, prove or give a counterexample.)