2025 Algebra 6112 Qualifying Exam
Instructions

Enter your name.# on the roster sheet together with a code name for yourself that is different from any code name that has already been entered.

Answer each question on a separate sheet of paper, and write your code name and the problem number on each sheet of paper that you submit for grading. Do not put your real name on any sheet of paper that you submit for grading.

Answer as many questions as you can. Do not use theorems which make the solution to the problem trivial. Always clearly display your reasoning. The judgment that you use in this respect is an important part of the exam.

This is a closed book, closed notes exam.

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1. Let R be a ring with identity and N a (left) R-module. Show that the tensor functor $- \otimes_R N$ from **mod**-R (the category of right R-modules) to $\mathbb{Z}$-**mod** (abelian groups) is right exact but not left exact.

2. Let R be a commutative ring with 1 and let $S \subset R$ be a multiplicatively closed set. Prove that $M \mapsto S^{-1}M$ is an exact functor from (R-mod) to ($S^{-1}R$-mod).

3. Recall the definition of $\mathrm{Ext}^k_R(M, N)$ for two R–modules M and N. Take a long exact sequence (possibly infinite) of projective modules P_i $i = 0, 1, \ldots$:

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

   and then consider the chain complex:

$$0 \to \mathrm{Hom}_R(P_0, N) \xrightarrow{d_0} \mathrm{Hom}(P_1, N) \xrightarrow{d_1} \mathrm{Hom}_R(P_2, N) \to \cdots$$

   Then $\mathrm{Ext}^k_R(M, N) := \mathrm{Ker}(d_k)/\,\mathrm{Im}(d_{k-1})$. Prove that

   (a) $\mathrm{Ext}^k_R(M, N)$ is well–defined. That is, the construction above is independent of the projective resolution of M chosen.

   (b) $\mathrm{Ext}^0_R(M, N) = \mathrm{Hom}_R(M, N)$.

   (c) Compute $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ where m, n are two positive integers.

4. Let K/k be an algebraic extension, and L an algebraically closed field. Given an embedding $\sigma\colon k \hookrightarrow L$, show that σ can be extended to an embedding $K \hookrightarrow L$.

5. Let $\mathbf{F}_p$ be a finite field with p elements. Determine the splitting field K of the polynomial $f(x) = x^p - x - 1$ over $\mathbf{F}_p$, and its Galois group $\mathrm{Gal}(K/\mathbf{F}_p)$.