Each question will be graded out of 10 points.

Problem 1. Suppose μ, ν are two positive measures on a measurable space $(X, \mathcal{M})$. Prove that when μ is **finite**, the following are equivalent:

(1) For $E \in \mathcal{M}$, $\nu(E) = 0$ implies $\mu(E) = 0$.
(2) For every $\varepsilon > 0$, there is a $\delta > 0$ such that $E \in \mathcal{M}$ and $\nu(E) < \delta$ implies $\mu(E) < \varepsilon$.

Problem 2. Suppose μ is a positive measure on $(X, \mathcal{M})$ and $f \in L^1(X, \mathcal{M}, \mu)$.

(1) Prove that if $E \subset X$ with $\mu(E) = 0$, then $\int_E f\, d\mu = 0$.
(2) Prove that if $\int_E f\, d\mu = 0$ for all $E \in \mathcal{M}$, then $f = 0$ μ-a.e.

Problem 3. Suppose ν is a **complex** measure on $(X, \mathcal{M})$. Show that for $f \in L^1(X, \mathcal{M}, \nu)$,

$$\left| \int f d\nu \right| \leq \int |f| d|\nu|.$$

Problem 4. Show that $F : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous (there is an $M > 0$ such that $|F(x) - F(y)| \leq M|x - y|$ for all $x, y \in \mathbb{R}$) if and only if F is absolutely continuous and $|F'| \leq M$ a.e.

Problem 5. Suppose $f : [0, 1] \to \mathbb{R}$ is continuous such that

$$\int_0^1 x^n f(x)\, dx = 0$$

for all $n \geq 2026$ where the above integral is the Riemann integral. Prove that $f = 0$.

Problem 6. Suppose F, X are Banach spaces with F finite dimensional. Show that a linear map $\varphi : X \to F$ is countinuous if and only if $\ker(\varphi)$ is closed.